From critical points to extended topological field theories

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Outline

I. From dynamical systems to monoidal bicategories

II. Extended topological quantum field theories

Dynamical systems

A general non-linear dynamical system is given by a system of DEs

$$\left.\begin{aligned} \dot{x}_1 &= F_1(x_1, \dots, x_n) \\ \dot{x}_2 &= F_2(x_1, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= F_n(x_1, \dots, x_n) \end{aligned}\right\} \qquad \begin{aligned} \dot{\underline{x}} &= F(\underline{x}) \\ F &: \mathbb{R}^n \longrightarrow \mathbb{R}^n \end{aligned}$$

An important class of dynamical systems are those which are conservative, in the sense that there is a scalar potential $f : U \longrightarrow \mathbb{R}$ with $U \subseteq \mathbb{R}^n$,

$$F = \nabla f$$

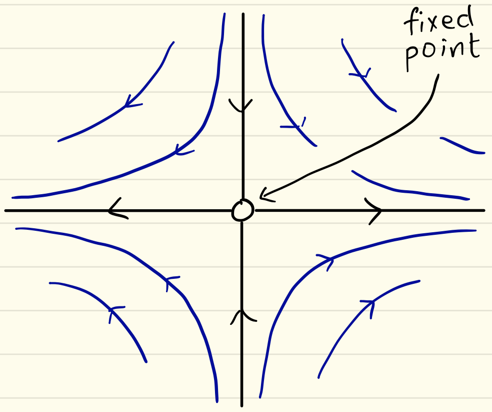
$$\{\text{fixed points of system}\} = \{\text{critical points of } f\}$$

$$\nabla f(\underline{x}) = 0$$

__Example__ Consider the system

$$\dot{x}_1 = x_1 \qquad\qquad \dot{\underline{x}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \underline{x}$$
$$\dot{x}_2 = -x_2$$

Solution trajectories look like $\underline{x}(t) = (Ae^{t}, Be^{-t})$ for any $A, B \in \mathbb{R}$.



Phase portrait

The scalar potential governing this system is

$$f = \tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_2^2$$

$$\nabla f = (x_1, x_2)$$

$$H_f = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Dynamical systems

To understand the dynamics near an isolated critical point of f we need to analyse the Hessian of f, i.e.

$$H_f := \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{1 \le i,j \le n},$$

its eigenvectors and eigenvalues. Actually the right way to think of this data is as a symmetric bilinear form on the tangent space $T_{\underline{x}^*}U$ at a critical point $\underline{x}^* \in U$, i.e.

$$\left(T_{\underline{x}^*}U, \langle , \rangle \right) \quad \text{where} \quad \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle = \left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{\underline{x} = \underline{x}^*}$$

with $\underline{u} = \underline{x} - \underline{x}^*$,

$$\underbrace{\dot{\underline{u}} = H_f\big|_{\underline{x}^*} \underline{u}}_{\text{linear system}} + \text{quadratic terms in } \underline{u} \text{ involving higher derivatives of } f$$

Morse Lemma If $H_f|_{\underline{x}^*}$ is invertible (i.e. the corresponding bilinear form is nondegenerate) for an isolated critical pt. $\underline{x}^*$ then there is a coordinate neighborhood around $\underline{x}^*$ where

$$f = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2$$

so that in those coordinates

$$H_f|_{\underline{x}^*} = \begin{array}{c} \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix} \\ \underbrace{\hphantom{xxxxxx}}_{p}\;\underbrace{\hphantom{xxxxxx}}_{q} \end{array}$$

Def^n A critical point $\underline{x}^*$ is <u>nondegenerate</u> if $H_f|_{\underline{x}^*}$ is invertible.

$\therefore$ locally $\quad \dot{\underline{u}} = H_f|_{\underline{x}^*}\, \underline{u} \qquad \underline{u} = \underline{x} - \underline{x}^*$

Quadratic spaces

Defⁿ The category $\mathcal{Q}$ of quadratic spaces over $\mathbb{R}$ has

- objects are f.d. vector spaces equipped with a nondegenerate symmetric bilinear form.
- morphisms $Q(V,W) = \{ T: V \to W \text{ linear} \mid \langle Tu, Tv\rangle = \langle u,v\rangle \ \forall u,v \}$.

Example
- $X_{p,q} := \left(\mathbb{R}^{\oplus p} \oplus \mathbb{R}^{\oplus q}, \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \right)$ is a representative set of objects (Sylvester's law of inertia)
- $X_{1,0} = (\mathbb{R}, (1)) \xrightarrow{\binom{1}{0}} \left(\mathbb{R} \oplus \mathbb{R}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) = X_{1,1}$ is a morphism.
- $(T_{\underline{x}^*} U, \langle , \rangle)$ $\quad \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle = \left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{\underline{x} = \underline{x}^*}$ at a nondeg. critical pt. $\underline{x}^*$.

Lemma $\mathcal{Q}$ is a symmetric monoidal category under direct sum of v spaces.

Clifford algebras

Associated to each quadratic space V is an algebra $C(V)$, the Clifford algebra which is universal among $\mathbb{R}$-algebras C (associative and unital) equipped with a linear map $\iota : V \longrightarrow C$ satisfying

$$\iota(v)\iota(w) + \iota(w)\iota(v) = 2\langle v,w\rangle \cdot 1_C .$$

(so e.g. $\iota(v)^2 = \langle v,v\rangle \cdot 1_C$)

This thing exists, is naturally $\mathbb{Z}_2$-graded, $V \hookrightarrow C(V)^1$ is injective and $C(V)$ is $2^{\dim(V)}$ dimensional.

Examples $C(X_{0,0}) \cong \mathbb{R}$, $C(X_{0,1}) \cong \mathbb{C}$, $C(X_{0,2}) \cong \mathbb{H}$

Lemma $C(-)$ is a strong monoidal functor $Q \longrightarrow \mathrm{Alg}_{\mathbb{R}}^{\mathbb{Z}_2}$, i.e. there are natural isomorphisms $C(0) \cong \mathbb{R}$ and

$$C(\underbrace{V \otimes W}_{\text{really direct sum!}}) \cong C(V) \otimes_{\mathbb{R}} C(W).$$

critical point $\underline{x}^*$ of f $\rightsquigarrow$ quadratic space $(T_{\underline{x}^*}U, Hf|_{\underline{x}^*})$

$\rightsquigarrow$ Clifford algebra $C(T_{\underline{x}^*}U, Hf|_{\underline{x}^*})$

$\rightsquigarrow$ Abelian category $\mathrm{Mod}^{\mathbb{Z}_2} C(T_{\underline{x}^*}U, Hf|_{\underline{x}^*})$.

finite-dimensional $\mathbb{Z}_2$-graded modules

$\underline{\text{Def}^n}$ Nondegenerate isolated critical points form a bicategory $\mathcal{C}rit_{\mathbb{R}}^{ndg}$

- $\underline{\text{objects}}$ quadratic spaces V
- $\underline{\text{1-morphisms}}$ $V \longrightarrow W$ are $\mathbb{Z}_2$-graded finite-dimensional $C(W)-C(V)$-bimodules.
- $\underline{\text{2-morphisms}}$ are bimodule homomorphisms.

$\underline{\text{Proposition}}$ $\mathcal{C}rit_{\mathbb{R}}^{ndg}$ is a symmetric monoidal bicategory in which every object is fully dualisable. (duals for objects and 1-morphisms)

$\underline{\text{Example}}$

- $\mathcal{C}rit_{\mathbb{R}}^{ndg}(0, V) = \mathrm{Mod}^{\mathbb{Z}_2} C(V)$. $\quad (0 = X_{0,0} = \mathbb{1})$
- $X_{0,1}^{\otimes 8} \cong \mathbb{1}$ (Bott periodicity)

Def^n A <u>bicategory</u> $\mathcal{B}$ consists of

- a class of objects $a, b, c, \ldots$
- for each pair a, b of objects a category $\mathcal{B}(a,b)$, objects of which are called <u>1-morphisms</u> and denoted $X : a \longrightarrow b$, and morphisms of which are called <u>2-morphisms</u>.
- a composition functor for objects a, b, c

$$\mathcal{B}(b,c) \times \mathcal{B}(a,b) \longrightarrow \mathcal{B}(a,c).$$

horizontal composition

$$(Y : b \to c,\ X : a \to b) \longmapsto (Y \circ X : a \longrightarrow c)$$

- unit 1-morphisms $1_a : a \longrightarrow a$ for each object a
- 2-isomorphisms "unitors", "associators"

satisfying some coherence conditions (same as for monoidal categories)

Def^n Let $\mathcal{B}, \mathcal{C}$ be bicategories. A <u>2-functor</u> $F : \mathcal{B} \longrightarrow \mathcal{C}$ is

- a function on objects $a \longmapsto F(a)$
- functors $\mathcal{B}(a,b) \longrightarrow \mathcal{C}(Fa, Fb)$
- natural isomorphisms

$$F(Y) \circ F(X) \cong F(Y \circ X)$$

$$1_{Fa} \cong F(1_a)$$

making some coherence diagrams commute.

<u>Example</u> If $\mathcal{B}$ is a bicategory, $\mathcal{B}(a,-) : \mathcal{B} \longrightarrow \underline{\mathcal{C}at}$ is a 2-functor, where $\underline{\mathcal{C}at}$ denotes small categories, functors and natural transformations.

<u>Def^n</u> Let $\mathcal{B}, \mathcal{C}$ be bicategories, $F, G : \mathcal{B} \longrightarrow \mathcal{C}$ 2-functors. A <u>pseudonatural transformation</u> $\varphi : F \longrightarrow G$ is

- a family of 1-morphisms $\{ \varphi_a : Fa \longrightarrow Ga \}_{a \in ob(\mathcal{B})}$
- for each $X : a \longrightarrow b$ in $\mathcal{B}$ a 2-isomorphism

$$\begin{tikzcd} Fa \arrow[r, "FX"] \arrow[d, "\varphi_a"'] \arrow[dr, phantom, "\varphi_X \, \nearrow"] & Fb \arrow[d, "\varphi_b"] \\ Ga \arrow[r, "GX"'] & Gb \end{tikzcd} \qquad GX \circ \varphi_a \overset{\varphi_X}{\cong} \varphi_b \circ FX$$

subject to coherence conditions.

<u>Def^n</u> (notation as above) Given pseudonatural transformations $\varphi, \psi : F \longrightarrow G$ a <u>modification</u> $q : \varphi \longrightarrow \psi$ is a family of 2-morphisms $\{ q_a : \varphi_a \longrightarrow \psi_a \}_a$ satisfying a condition (omitted).

Lemma Let $\mathcal{B}, \mathcal{C}$ be bicategories, with $\mathcal{B}$ small. Then there is a bicategory

$$\mathrm{Bicat}(\mathcal{B},\mathcal{C}) \left\{ \begin{array}{ll} \underline{\text{objects}} & \text{2-functors} \\[2ex] \underline{\text{1-morphisms}} & \text{pseudonatural transformations} \\[2ex] \underline{\text{2-morphisms}} & \text{modifications} \end{array} \right.$$

Monoidal bicategory (rough version) is a bicategory $\mathcal{B}$ with

- tensor for objects $(a,b) \longmapsto a \,\square\, b$
- tensor for 1- and 2-morphisms, via a functor

$$\mathcal{B}(a_1,a_2) \times \mathcal{B}(b_1,b_2) \longrightarrow \mathcal{B}(a_1 \square b_1,\ a_2 \square b_2)$$

- associators, unitors, coherence.

Def^n A *monoidal bicategory* is a bicategory $\mathcal{B}$ equipped with

- a 2-functor $\square : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B}$
- an adjoint equivalence α in $\mathrm{Bicat}((\mathcal{B}\times\mathcal{B})\times\mathcal{B}, \mathcal{B})$ between the two legs of the following diagram (the *associator*)

$$\begin{tikzcd}
(\mathcal{B}\times\mathcal{B})\times\mathcal{B} \arrow[rr, "\text{rebracket}"] \arrow[d, "\square\times 1"'] & & \mathcal{B}\times(\mathcal{B}\times\mathcal{B}) \arrow[d, "1\times\square"] \\
\mathcal{B}\times\mathcal{B} \arrow[dr, "\square"'] \arrow[rr, phantom, "\alpha \Nearrow"] & & \mathcal{B}\times\mathcal{B} \arrow[dl, "\square"] \\
& \mathcal{B} &
\end{tikzcd}$$

$\leftarrow$ a 2-functor

i.e. $a \square (b \square c) \overset{\alpha}{\cong} (a \square b) \square c$

i.e. α is a pseudonatural transformation.

- an invertible modification π, the pentagonator

$$\begin{tikzcd}
& (A \square B) \square (C \square D) \arrow[dr, "\alpha"] & \\
((A \square B) \square C) \square D \arrow[ur, "\alpha"] \arrow[d, "\alpha"'] \arrow[rr, phantom, "\Uparrow \pi"] & & A \square (B \square (C \square D)) \\
(A \square (B \square C)) \square D \arrow[dr, "\alpha"'] & & \\
& A \square ((B \square C) \square D) \arrow[uur, "\alpha"'] &
\end{tikzcd}$$

⌜each vertex is a 2-functor
each edge a transformation⌟

+ units, unitors and lots of coherence!

Def^n A *symmetric monoidal bicategory* is a monoidal bicategory $\mathcal{B}$ with

- an adjoint equivalence β in $\mathrm{Bicat}(\mathcal{B}\times\mathcal{B}, \mathcal{B})$ between the legs of

$$\begin{tikzcd} \mathcal{B}\times\mathcal{B} \arrow[rr, "\square"] \arrow[dr, "swap"'] & \arrow[d, phantom, "\Downarrow \beta"] & \mathcal{B} \\ & \mathcal{B}\times\mathcal{B} \arrow[ur, "\square"'] & \end{tikzcd}$$

i.e. $a \square b \overset{\beta}{\cong} b \square a$

- an invertible modification called *syllepsis*

$$\begin{tikzcd} a\square b \arrow[rr, "1_{a\square b}"] \arrow[dr, "\beta"'] & \arrow[d, phantom, "\Downarrow \mathfrak{z}"] & a\square b \\ & b\square a \arrow[ur, "\beta"'] & \end{tikzcd}$$

- invertible modifications relating β and the associator + coherence

<u>Examples</u> ① $(\underline{Cat}, \times)$ categories, functors, natural transformations, Cartesian product

② $(\underline{Alg}_k, \otimes_k)$ algebras, bimodules, bimodule maps, tensor.

③ $(Crit^{ndg}_{\mathbb{R}}, \oplus)$ quadratic spaces, Clifford bimodules and maps, direct sum.

<u>References</u> (not a historical survey!)

- Chris Schommer-Pries' PhD thesis
- Nick Gurski "Loop spaces, and coherence for monoidal and braided monoidal bicategories".
- P. Pstragowski "On dualizable objects in monoidal bicategories, framed surfaces and the Cobordism Hypothesis" PhD thesis.

Duals in symmetric monoidal bicategories

Let $\mathcal{B}$ be a monoidal bicategory. A right dual to an object a is a^* and 1-morphisms $ev_a : a \Box a^* \longrightarrow \mathbb{1}$, $coev_a : \mathbb{1} \longrightarrow a^* \Box a$ and cusp isomorphisms in $\mathcal{B}(a,a)$

$$\begin{tikzcd}
a \arrow[r, "\cong"] \arrow[rrrrr, bend right, "1_a"'] & a \Box \mathbb{1} \arrow[r, "1 \Box coev_a"] & a \Box (a^* \Box a) \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\Downarrow"] & (a \Box a^*) \Box a \arrow[r, "ev_a \Box 1"] & \mathbb{1} \Box a \arrow[r, "\cong"] & a \\
 & & {} & & &
\end{tikzcd}$$

$$\begin{tikzcd}
a^* \arrow[r, "\cong"] \arrow[rrrrr, bend right, "1_{a^*}"'] & \mathbb{1} \Box a^* \arrow[r, phantom, ""] & (a^* \Box a) \Box a^* \arrow[r, phantom, "\cong"] \arrow[d, phantom, "\Downarrow"] & a^* \Box (a \Box a^*) \arrow[r, "1 \Box ev_a"] & a^* \Box \mathbb{1} \arrow[r] & a^* \\
 & & {} & & &
\end{tikzcd}$$

Lemma In a symmetric monoidal category every right dual is also a left dual.

Duals in symmetric monoidal bicategories

<u>Def^n</u> Let $\mathcal{B}$ be a symmetric monoidal bicategory. An object a is <u>fully dualisable</u> if it has a dual object such that both ev_a and $coev_a$ have both left and right adjoints.

- Every object in $\mathrm{Crit}^{ndg}_{\mathbb{R}}$ is fully dualisable $(V, B)^* := (V, -B)$.

<u>Theorem</u> (Pstragowski) [2D cobordism hypothesis] There is an equivalence

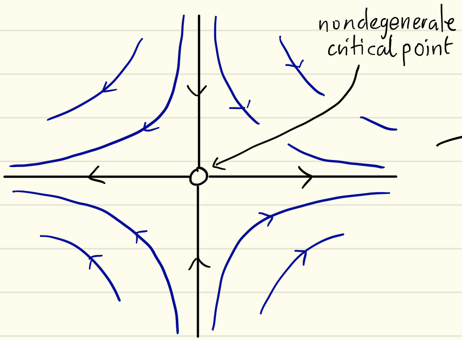
$$\mathrm{Bicat}_{sym.mon}(\mathrm{Bord}_2^{fr}, \mathcal{B}) \cong K(\mathcal{B}^{fd})$$

fully dualisable objects

framed bordism bicategory

core, i.e. keep equivalences and 2-isomorphisms

$$f = \tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_2^2$$

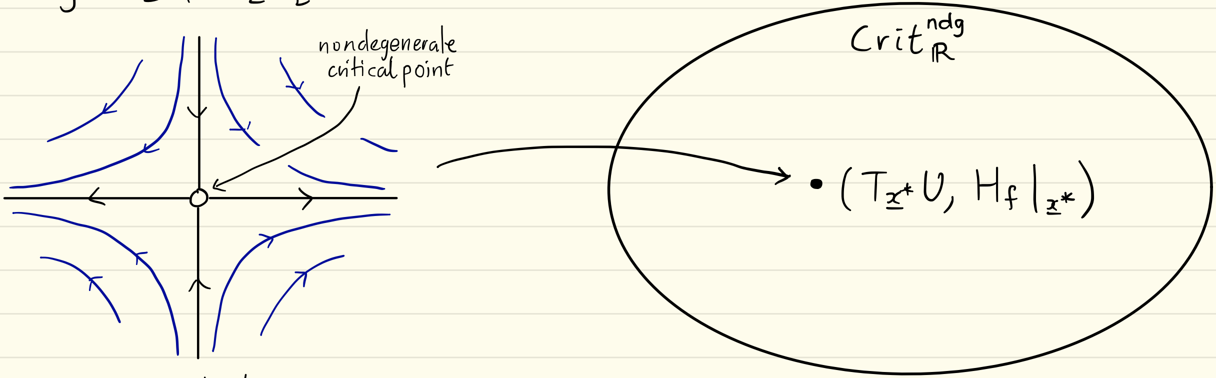


Phase portrait

$\mathrm{Crit}^{\mathrm{ndg}}_{\mathbb{R}}$

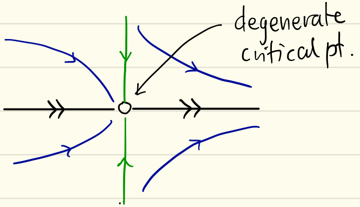
$\bullet\ (T_{\underline{x}^*}U,\ H_f|_{\underline{x}^*})$

$$f = \tfrac{1}{2} x_1^2 - \tfrac{1}{2} x_2^2$$



Phase portrait

$$f = \tfrac{1}{3} x_1^3 - \tfrac{1}{2} x_2^2$$



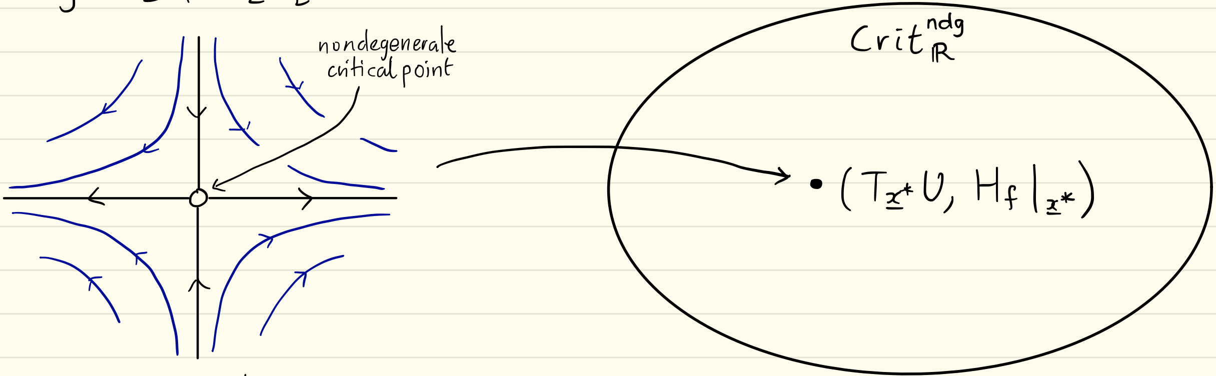
$$\dot{x}_1 = x_1^2$$
$$\dot{x}_2 = -x_2$$

Around an isolated (degenerate) critical point $\underline{x}^*$

$$\underbrace{\dot{\underline{u}} = H_f|_{\underline{x}^*} \underline{u}}_{\text{linear system}} + \text{quadratic terms in } \underline{u} \text{ involving higher derivatives of the potential } f.$$

where $\underline{u} = \underline{x} - \underline{x}^*$, the dynamics <u>do</u> depend on the higher derivatives of f.

$$f = \tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_2^2$$



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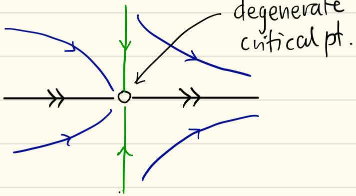

$$\dot{x}_1 = x_1^2$$
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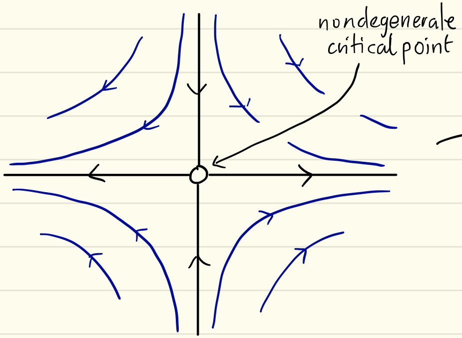
$$\underbrace{\dot{\underline{u}} = H_f|_{\underline{x}^*}\,\underline{u}}_{\text{linear system}} + \text{quadratic terms in } \underline{u} \text{ involving higher derivatives of the potential } f.$$

<u>Question</u> What algebra to associate to $(f, \underline{x}^*)$?

- reduce to $C(T_{\underline{x}^*}U, H_f|_{\underline{x}^*})$ in the nondeg. case
- form a symmetric monoidal bicategory

$$f = \tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_2^2$$



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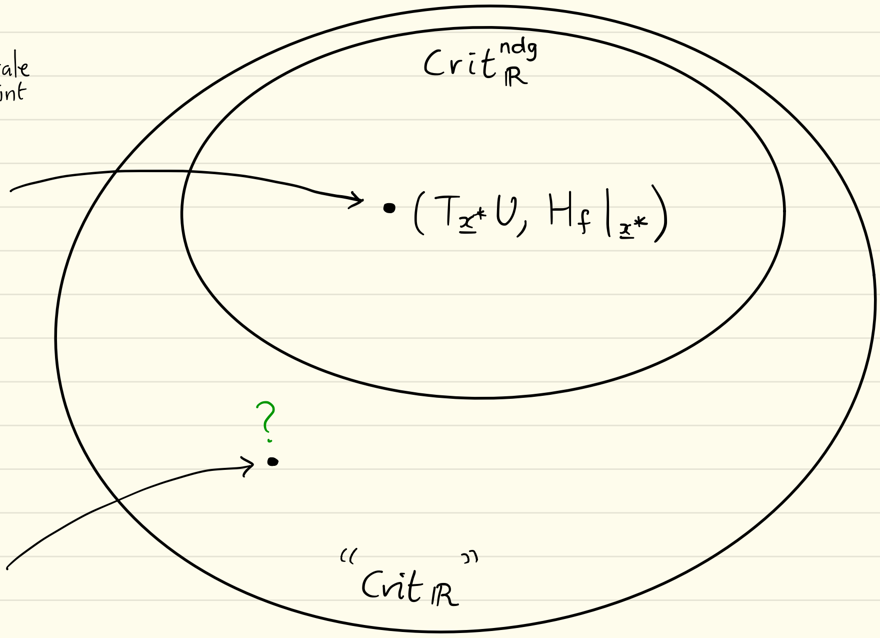

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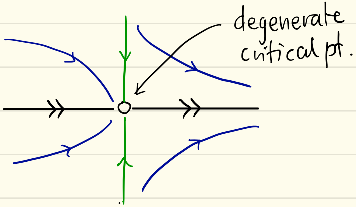

$$\dot{x}_1 = x_1^2$$
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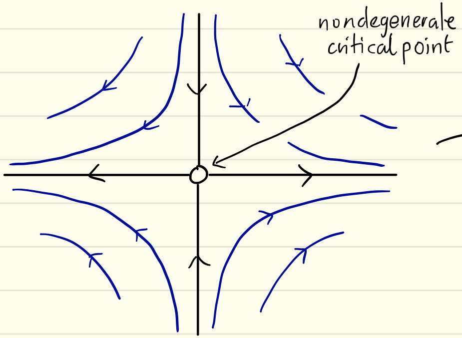
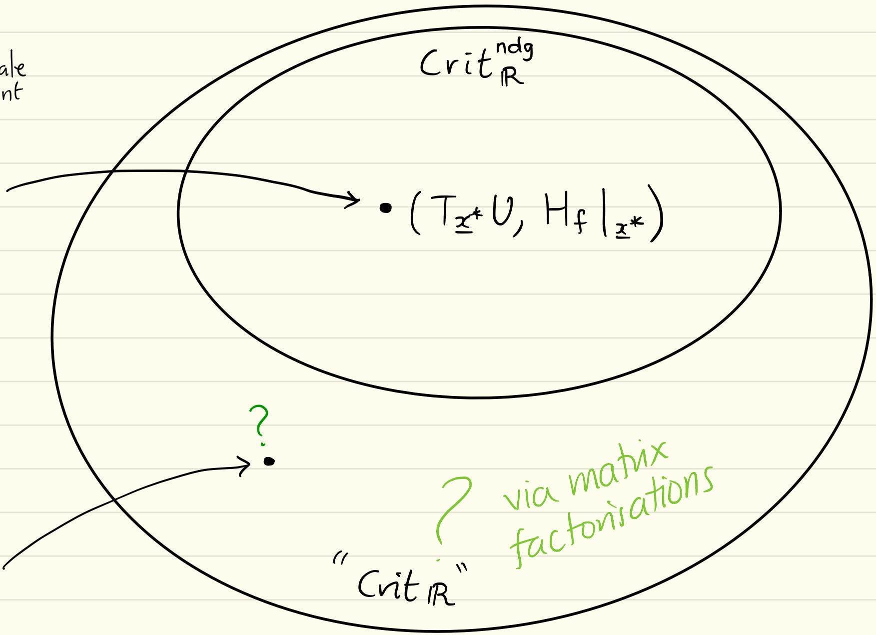


Phase portrait



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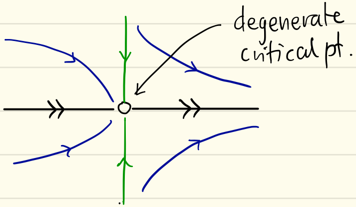

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<u>Question</u> What algebra to associate to $(f, \underline{x}^*)$?

- reduce to $C(T_{\underline{x}^*}U, H_f|_{\underline{x}^*})$ in the nondeg. case
- form a symmetric monoidal bicategory

Matrix factorisations

Let X be a $\mathbb{Z}_2$-graded f.d. module over the Clifford algebra

$C(X_{p,q})$: generated by $\gamma_1, \dots, \gamma_{p+q}$ subject to

$$\gamma_1^2 = \cdots = \gamma_p^2 = 1$$
$$\gamma_{p+1}^2 = \cdots = \gamma_{p+q}^2 = -1$$
$$\gamma_i \gamma_j + \gamma_j \gamma_i = 0 \qquad i \neq j$$

Matrix factorisations

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$$\gamma_i \gamma_j + \gamma_j \gamma_i = 0 \qquad i \neq j$$

Dirac's idea

Set $A = \mathbb{R}[x_1, \dots, x_{p+q}]$, and

$$X \otimes_{\mathbb{R}} A \circlearrowleft \partial = \sum_{i=1}^{n} x_i \gamma_i$$

$\mathbb{Z}_2$-graded free A-module

$$\partial^2 = \sum_{i,j} x_i x_j \gamma_i \gamma_j$$
$$= \sum_i x_i^2 \gamma_i^2$$
$$= \underbrace{x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2}_{\text{acting on } X \otimes_{\mathbb{R}} A}$$

Potentials Let k be a commutative $\mathbb{Q}$-algebra, then $f \in R = k[x_1, \dots, x_n]$ is called a potential if

(i) $\partial_{x_1} f, \dots, \partial_{x_n} f$ is quasi-regular

(ii) $R/(\partial_{x_1} f, \dots, \partial_{x_n} f)$ is a f.g. free k-module

(iii) the Koszul complex of $\partial_{x_1} f, \dots, \partial_{x_n} f$ is exact outside deg. 0.

Example $f \in \mathbb{C}[x_1, \dots, x_n]$ such that $\dim_{\mathbb{C}} \mathbb{C}[x_1, \dots, x_n]/(\partial_{x_1} f, \dots, \partial_{x_n} f) < \infty$.

(isolated critical points)

Def^n The DG-category $\mathcal{A} = mf(R, f)$ has

- objects f. rank matrix factorisations of f, i.e. $X \ni d_X^2 = f \cdot 1_X$.
- morphisms $\mathcal{A}(X, Y) = \left(\mathrm{Hom}_R(X, Y),\ \alpha \mapsto d_Y \alpha - (-1)^{|\alpha|} \alpha d_X \right)$.

This is a $\mathbb{Z}_2$-graded DG-category over R.

<u>Remarks</u>

- $hmf(R,f) := H^0 mf(R,f)$ is triangulated (Calabi-Yau).

- Given a quadratic space V with associated quadratic $f \in Sym(V^*)$

$$\mathrm{Mod}^{\mathbb{Z}_2}_{f.d.} C(V) \cong hmf(Sym(V^*), f)^{\omega}$$

(Buchweitz–Eisenbud–Herzog)

<u>Remarks</u>

- $\mathrm{hmf}(R,f) := H^0\mathrm{mf}(R,f)$ is triangulated (Calabi-Yau).
- Given a quadratic space V with associated quadratic $f \in \mathrm{Sym}(V^*)$

$$\mathrm{Mod}^{\mathbb{Z}_2}_{f.d.}\, C(V) \cong \mathrm{hmf}(\mathrm{Sym}(V^*), f)^{\omega}$$

(Buchweitz–Eisenbud–Herzog)

From a potential f to an A_∞-algebra A_f

Assume k is a field and $\mathrm{Sing}(f) = \{0\}$. Then there is a standard generator

$$\mathrm{thick}(G) = \mathrm{hmf}(R,f)^{\omega}$$

$$\mathrm{perf}\,\mathrm{End}_R(G) \cong \mathrm{hmf}(R,f)^{\omega} \qquad \text{(Keller–Lefevre)}$$

A_∞-transfer (minimal model theorem)

$$\mathrm{perf}_\infty \underbrace{H^*\mathrm{End}_R(G)}_{} \cong \mathrm{hmf}(R,f)^{\omega}$$

A_∞-algebra A_f, is a Clifford algebra for quadratic f.
A_∞-products package higher derivatives of f.

<u>Pseudo-def^n</u> $\mathcal{C}rit_{\mathbb{R}}$ is the bicategory of A_∞-algebras $A_{(f,\underline{x}^*)}$ associated to isolated critical points, A_∞-bimodules and A_∞-bimodule maps.

<u>Theorem</u> (Carqueville–Montoya '18) $\mathcal{C}rit_{\mathbb{R}}$ is a symmetric monoidal bicategory in which every object is fully dualisable, and therefore determines an extended 2D framed TFT

$$\mathrm{Bord}_2^{fr} \longrightarrow \mathcal{C}rit_{\mathbb{R}}.$$

Moreover $\mathcal{C}rit_{\mathbb{R}}^{ndg} \subset \mathcal{C}rit_{\mathbb{R}}$.

essentially due to Buchweitz–Eisenbud–Herzog.

Sketch of $\mathcal{LG}_k$ (e.g. $\mathcal{C}rit_{\mathbb{R}}$) (bicategory of Landau-Ginzburg models)

k any commutative ring

$$
\begin{tikzcd}
 & (\underline{y}, V(\underline{y})) \arrow[dr, "G"] & \\
(\underline{x}, W(\underline{x})) \arrow[ur, bend left, "E"] \arrow[ur, bend right, "F"'] \arrow[ur, phantom, "\Downarrow"] \arrow[rr, "G \otimes_{\mathbb{R}[\underline{y}]} E"'] & & (\underline{z}, U(\underline{z}))
\end{tikzcd}
$$

$$W \,\square\, V = W + V$$

$$\mathbb{1} = (\phi, 0)$$

$$W^* = -W$$

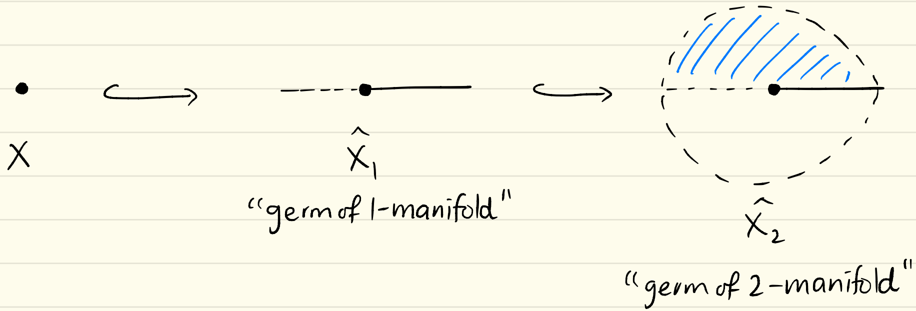
Reference N. Carqueville, DM "Adjoints and defects in Landau-Ginzburg models"

Brief sketch of Bord_2^{fr} (following Schommer-Pries, Pstragowski)

Def^n Let X^k be a manifold, possibly with corners. If $k < 2$ a <u>2-halo</u> over X is a sequence of inclusions of pro-manifolds

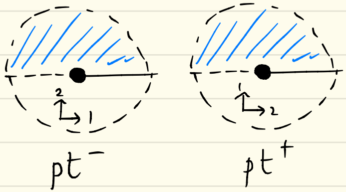
$$X \subseteq \hat{X}_1 \subseteq \hat{X}_2$$

such that $X \subseteq \hat{X}_1$, $X \subseteq \hat{X}_2$ have the structure of cooriented halations of dim 1, 2 respectively.



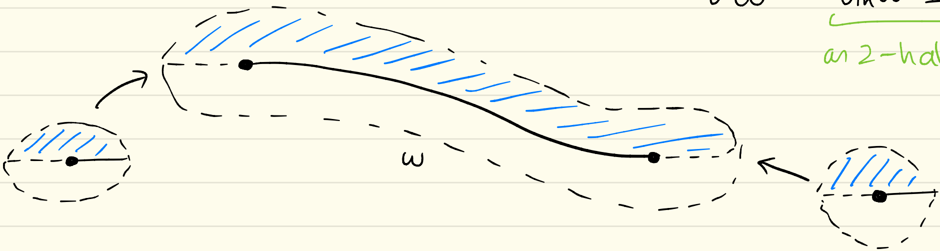
Theorem (Schommer-Pries) There is a symmetric monoidal bicategory Bord_2^{fr}

objects framed 2-haloed 0-manifolds

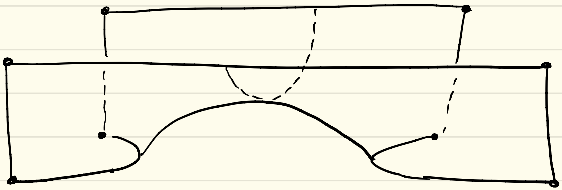


1-morphisms framed 2-haloed 1-bordisms

$$\partial\omega \cong \underbrace{\partial_{in}\omega \amalg \partial_{out}\omega}_{\text{an 2-haloed 0-mfld}}$$



2-morphisms framed 2-haloed 2-bordisms $/ \cong$



$$\begin{tikzcd} A \arrow[r, bend left, "v"] \arrow[r, bend right, "\omega"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

$\partial\alpha = \partial_0 \cup \partial_m \qquad \partial_0 \cap \partial_m = \partial\partial_0$
$\qquad\qquad\qquad\qquad\qquad\qquad = \partial\partial_m$

$\partial_m \cong \omega \amalg v$

$\partial_0 \cong A \times I \amalg B \times I$

Structure of $\mathcal{LG}_k$ under control $\Longrightarrow$ one can actually compute this TQFT

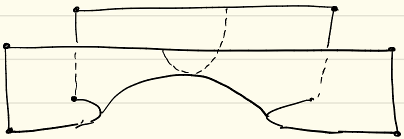
$$\mathrm{Bicat}_{\mathrm{sym.mon}}\left(\mathrm{Bord}_2^{fr}, \mathcal{LG}_k\right) \simeq K\left(\mathcal{LG}_k^{fd}\right)$$

Application The "TQFT with corners" constructed by Khovanov and Rozansky can be derived / corrected using the cobordism hypothesis as extended TQFTs

$$pt^+ \longmapsto x^{N+1} \in \mathcal{LG}_R .$$

p. 89 Montoya's thesis

Proving this uses explicit formulas for ev, coev in $\mathcal{LG}_k$.



$$\longmapsto \quad \begin{tikzcd}[column sep=huge] y^{N+1} \boxtimes (x^{N+1})^* \arrow[r, bend left, "1 \boxtimes 1 = 1"] \arrow[r, bend right, "\mathrm{coev} \circ \mathrm{ev}"'] \arrow[r, phantom, "\Downarrow"] & y^{N+1} \boxtimes (x^{N+1})^* \end{tikzcd}$$

$$y^{N+1} \boxtimes (x^{N+1})^* = y^{N+1} - x^{N+1}$$