Tour of well-generated triangulated categories

The theory of well-generated triangulated categories is one of Neeman's most influential contributions, and it is a theory which continues to become of increasing relevance to other areas of mathematics as the role of "unbounded" triangulated categories widens. This talk aims to give a tour of theory, emphasising Neeman's original point of view.

Short version: $D(\mathcal{A})$, $\mathcal{A}$ Grothendieck abelian, is always well-generated but not always compactly generated.
↖ Brown representability still holds.

Long version:
1. Introduction / Def$^{\text{n}}$s
2. Examples
3. Analogy to Grothendieck ab. cats.

1 Introduction

We begin life caring about additive functors between abelian categories

$$\mathrm{Sh}(X) \xrightarrow{\mathrm{Hom}(\mathcal{F},-)} \underline{Ab},$$

and that life is good. However at some point we are ejected from the Eden of abelian categories into the harsh world of injective resolutions and spectral sequences, ameliorated only by the language of triangulated functors between triangulated categories

$$\mathbb{D}(\mathrm{Sh}(X)) \xrightarrow{\mathbb{R}\mathrm{Hom}(\mathcal{F},-)} \mathbb{D}(\underline{Ab}).$$

However the memory of Eden haunts us, and we try to return

$$
\begin{tikzcd}
 & D(Sh(X)) \arrow[rr, "\mathbb{R}Hom(\mathcal{F},-)"] \arrow[d, equal] \arrow[drr, phantom, "\circlearrowleft \text{ def}"] & & D(\underline{Ab}) \arrow[d, equal] & \\
 & \mathcal{T} \arrow[rr, "F"] \arrow[dl, "H"'] \arrow[dd, "\text{Yoneda}"] & & \mathcal{S} \arrow[dr, "H"] \arrow[dd, "\text{Yoneda}"'] & \\
A(\mathcal{T}) \arrow[rrrr, dashed, "A(F)"] \arrow[dr, "\text{inc}"'] & & & & A(\mathcal{S}) \arrow[dl, "\text{inc}"] \\
 & Add(\mathcal{T}^{op}, \underline{Ab}) & & Add(\mathcal{S}^{op}, \underline{Ab}) &
\end{tikzcd}
$$

triangulated

universal homological functor

abelianisation subcat of f.p. objects (Freyd, Verdier)

abelian

The abelianisation $h_{\mathcal{T}}$ respects all structure on $\mathcal{T}$, in the sense that it sends

- triangles to long exact sequences
- coproducts to coproducts.

However, this $A(\mathcal{T})$ is not Eden, but rather a "terrible" haunted garden:

<u>Theorem</u> (Neeman) $\mathcal{T}(-,\mathbb{Z})$ has a proper class of subobjects in $A(\mathcal{T})$, $\mathcal{T} = D(\underline{Ab})$

<u>Goal</u> Rescue the dream, by finding a good "closed subset" of $A(\mathcal{T})$, i.e.

- a locally small abelian category $\mathcal{C}_{\mathcal{T}}$, with
- a quotient map $A(\mathcal{T}) \xrightarrow{\pi} \mathcal{C}_{\mathcal{T}}$ (identifying $\mathcal{C}_{\mathcal{T}}$ as $A(\mathcal{T})/\mathcal{B}$, $\mathcal{B}$ Serre and closed under products, s.t. π has a fully faithful left adjoint. Thin $D(X) \underset{\mathbb{R}\Gamma_Z}{\overset{\text{inc}}{\leftrightarrows}} D_Z(X)$)
- the composite $\mathcal{T} \longrightarrow A(\mathcal{T}) \longrightarrow \mathcal{C}_{\mathcal{T}}$ maps nonzero objects to nonzero objects, and preserve coproducts, and
- $\mathcal{T} \longmapsto \mathcal{C}_{\mathcal{T}}$ is functorial.

This goal is achieved by the theory of well-generated triangulated categories. Let me sketch how, before explaining definitions.

If $\mathcal{T}$ is <u>well-generated</u> then

- $\mathcal{T} = \bigcup_{\alpha \text{ regular}} \mathcal{T}^\alpha$, $\mathcal{T}^\alpha$ α-localising (triang. subcat closed under $< \alpha$ coproducts)
  Objects of $\mathcal{T}^\alpha$ are called <u>α-compact</u> $(\mathcal{T}^{\aleph_0} = \mathcal{T}^c)$.

- An abelian category satisfies $[AB5^\alpha]$ if it is $[AB4]$ (coproducts of monos are mono) and α-filtered colimits are exact.

- For α regular there is a universal coproduct-preserving homological functor

$$H_\alpha : \mathcal{T} \longrightarrow A_\alpha(\mathcal{T})$$

into an abelian category $A_\alpha(\mathcal{T})$ which is $[AB5^\alpha]$ (Neeman, Appendix B).

$$\begin{tikzcd}
 & \mathcal{T} \arrow[dl, "H"'] \arrow[d, "H_\alpha"] \arrow[dr, "h_\alpha"] & \\
A(\mathcal{T}) \arrow[r, dashed, bend right, "f_*"'] \arrow[r, phantom, "\perp"] & A_\alpha(\mathcal{T}) \arrow[l, dashed, bend right, "f^*"'] \arrow[r, dashed, "\cong"] & \mathrm{Add}_\alpha((\mathcal{T}^\alpha)^{op}, \underline{Ab})
\end{tikzcd}$$

(ref: Krause "Localization theory of triangulated categories")

$h_\alpha(X) = \mathcal{T}(-,X)|_{\mathcal{T}^\alpha}$

(locally small, h_α pres. coproducts, abelian with $[AB5^\alpha]$)

the f.f. left adjoint is closure of $\mathcal{T}(-,X)$, $X \in \mathcal{T}^\alpha$ under all colimits

$\boxed{\text{Upshot}}$ $\mathcal{C}_\mathcal{T} = A_\alpha(\mathcal{T}) \cong \mathrm{Add}_\alpha((\mathcal{T}^\alpha)^{op}, \underline{Ab})$ is what we are looking for (α sufficiently large. For $\mathcal{T} = \mathbb{D}(\mathbb{Z})$, $\alpha = \aleph_0$ will do). This is the "reasonable abelianisation" of $\mathcal{T}$.

From now on $\mathcal{T}$ is a triangulated category with arbitrary coproducts, small Hom sets, and α is a regular cardinal.

<u>Def^n</u> $X \in \mathcal{T}$ is <u>α-small</u> if every morphism into a coproduct

$$X \longrightarrow \coprod_{i \in I} Y_i$$

factors through a subcoproduct

$$X \longrightarrow \coprod_{i \in J} Y_i \longrightarrow \coprod_{i \in I} X_i$$

with $|J| < \alpha$. Let $\mathcal{T}^{(\alpha)} = \{ X \in \mathcal{T} \mid X \text{ is } \alpha\text{-small} \}$.

<u>Def^n</u> A class $T \subseteq \mathcal{T}$ is an <u>α-perfect class</u> if it contains 0, and

(i) Any morphism $t \longrightarrow \coprod_{i \in I} Y_i$ with $t \in T$, $Y_i \in \mathcal{T}$, $|I| < \alpha$ factors as

$$t \longrightarrow \coprod_{i \in I} t_i \xrightarrow{\amalg_i f_i} \coprod_{i \in I} Y_i$$

for some $t_i \in T$.

(ii) Given morphisms $\{ f_i : t_i \longrightarrow Y_i \}_{i \in I}$ with $t_i \in T$, $Y_i \in \mathcal{T}$, $|I| < \alpha$ if a composite of the form

$$t \longrightarrow \coprod_{i \in I} t_i \xrightarrow{\amalg_i f_i} \coprod_{i \in I} Y_i$$

vanishes then there is a factorisation $t_i \xrightarrow{g_i} s_i \longrightarrow Y_i$ of each f_i such that $t \longrightarrow \coprod_{i \in I} t_i \xrightarrow{\amalg g_i} \coprod_i s_i$ already vanishes.

<u>Definition</u> A triangulated subcategory $\mathcal{S} \subseteq \mathcal{T}$ is <u>α-localising</u> if it is closed under coproducts of $< \alpha$ of its objects.

and essentially small

<u>Theorem</u> (Neeman) Suppose $\mathcal{S} \subseteq \mathcal{T}$ is α-localising. The homological functor

$$\begin{aligned} \mathcal{T} &\longrightarrow \mathrm{Add}_\alpha(\mathcal{S}^{op}, \underline{Ab}) \\ t &\longmapsto \mathcal{T}(-,t)|_{\mathcal{S}} \end{aligned}$$

induces an exact functor

$$A(\mathcal{T}) \xrightarrow{\pi} \mathrm{Add}_\alpha(\mathcal{S}^{op}, \underline{Ab})$$

has products and coproducts, products as in $\mathcal{C}at(\mathcal{S}^{op}, \underline{Ab})$

which preserves coproducts if and only if

(i) every object of $\mathcal{S}$ is α-small,

(ii) $\mathcal{S}$ is α-perfect.

In this case π is a (ω) localisation (it has a left adjoint π_λ s.t. $1 \xrightarrow{\cong} \pi\pi_\lambda$).

<u>Def^n/Thm</u> There is a unique maximal α-perfect class $\mathcal{T}^\alpha \subseteq \mathcal{T}^{(\alpha)}$, and it is α-localising. The objects of $\mathcal{T}^\alpha$ are called <u>α-compact</u>.

<u>Note</u> $\mathcal{S} = \mathcal{T}^\alpha$ makes $\mathrm{Add}_\alpha(\mathcal{S}^{op}, \underline{Ab})$ as big as possible.

<u>Def^n</u> $\mathcal{T}$ is <u>α-compactly generated</u> if

- $\mathcal{T}^\alpha$ is essentially small (so $\mathrm{Add}_\alpha((\mathcal{T}^\alpha)^{op}, \underline{Ab})$ is a category)

- $\langle \mathcal{T}^\alpha \rangle = \mathcal{T}$ (so $\mathcal{T} \longrightarrow \mathrm{Add}_\alpha((\mathcal{T}^\alpha)^{op}, \underline{Ab})$ sends nonzero objects to nonzero objects)

If $\mathcal{T}$ is α-compactly generated for some α, it is <u>well-generated</u>.

2 Examples (due to Neeman unless indicated otherwise)

(a) Compactly generated $= \aleph_0$-compactly generated, $\mathcal{T}^{\aleph_0} = \mathcal{T}^c$.

(b) If $\mathcal{A}$ is Grothendieck abelian $D(\mathcal{A})$ is well-generated

c-gen $\begin{cases} \mathbb{D}(R), \; R \text{ a ring} \\ \mathbb{D}(\mathrm{Qcoh}\,X), \; X \text{ quasi-compact semi-separated} \end{cases}$

non-c-gen $\{ \; \mathbb{D}(Ab/M)$, M smooth manifold, connected non-compact

(c) $\mathbb{K}(\mathrm{Proj}\text{-}R)$, $\mathbb{K}(\mathrm{Inj}\text{-}R)$ are well-generated (compactly generated under finiteness hypotheses: Jørgensen, Iyengar-Krause, Neeman)

(d) $\mathbb{D}(R)^\alpha = \{ X \in \mathbb{D}(R) \mid \exists P \; \mathbb{K}\text{-projective}, \; \mathrm{rank}(P_i) < \alpha \text{ for all } i, \text{ and } X \overset{qis}{\cong} P \}$.

$\mathbb{D}(\mathrm{Qcoh}\,X)^\alpha = \{ X \in \mathbb{D}(\mathrm{Qcoh}\,X) \mid X \text{ is locally in } \mathbb{D}(\Gamma(U, \mathcal{O}_X))^\alpha \}$. (M)

(e) (Krause) Suppose $\mathcal{A}$ is Grothendieck abelian, $\mathcal{A}^\alpha$ abelian and generates $\mathcal{A}$. Then $\mathbb{D}(\mathcal{A})$ is α-compactly generated and $\mathbb{D}(\mathcal{A})^\alpha = \mathbb{D}(\mathcal{A}^\alpha)$.

($\mathcal{A}^\alpha = \alpha$-presentable objects, i.e. $\mathcal{A}(X,-)$ preserves α-filtered colimits)

<u>Remark</u> α-compactness is <u>local</u>. This is best understood using $A(\mathcal{T})$

- Two Bousfield subcategories $\mathcal{I}_1, \mathcal{I}_2 \subseteq \mathcal{T}$ intersect <u>properly</u> if

$$\mathcal{I}_1 * \mathcal{I}_2 = \mathcal{I}_2 * \mathcal{I}_1 \longleftarrow \{ X \mid \exists \text{ triangle } \underset{\mathcal{I}_2}{\underset{\Cap}{I_2}} \to X \to \underset{\mathcal{I}_1}{\underset{\Cap}{I_1}} \to \Sigma I_2 \}$$

- A <u>cocover</u> of $\mathcal{T}$ (think $\mathbb{D}_{Z_1}(X), \mathbb{D}_{Z_2}(X) \subseteq \mathbb{D}(X)$ with $Z_1 \cap Z_2 = \emptyset$ i.e. $Z_1^c \cup Z_2^c = X$) is a pair $\mathcal{I}_1, \mathcal{I}_2$ of properly intersecting Bousfield subcategories with $\mathcal{I}_1 \cap \mathcal{I}_2 = 0$.

- Suppose $\mathcal{T}/\mathcal{I}_i$ is α-compactly generated for $i \in \{1,2\}$ and the images of

$$\mathcal{I}_i \longrightarrow \mathcal{T} \xrightarrow{p_j} \mathcal{T}/\mathcal{I}_j \qquad j \neq i$$

are α-compactly generated in $\mathcal{T}/\mathcal{I}_j$.

<u>Claim</u> $\mathcal{T}$ is α-compactly generated and

call this $\mathcal{T}^{|\alpha|}$

$$\mathcal{T}^\alpha = \{ X \in \mathcal{T} \mid p_j(X) \in (\mathcal{T}/\mathcal{I}_j)^\alpha \quad j = 1,2 \}.$$

The idea of the proof is contained in the following picture

$\mathcal{T}^{|\alpha|} \longrightarrow (\mathcal{T}/\mathcal{I}_1)^\alpha$ is a Verdier quotient

$$\begin{tikzcd}
A(\mathcal{T}) \arrow[rrr, "\pi"] \arrow[d, "A(p_1)"'] \arrow[dr, "A(p_2)"] & & & \mathrm{Add}_\alpha((\mathcal{T}^{|\alpha|})^{op}, \underline{Ab}) \arrow[dl, "q_1"'] \arrow[d, "q_2"] \\
A(\mathcal{T}/\mathcal{I}_1) \arrow[rr, "\pi_1"] & A(\mathcal{T}/\mathcal{I}_2) \arrow[rr, "\pi_2"'] & \mathrm{Add}_\alpha((\mathcal{T}/\mathcal{I}_1)^\alpha, \underline{Ab}) & \mathrm{Add}_\alpha((\mathcal{T}/\mathcal{I}_2)^\alpha, \underline{Ab})
\end{tikzcd}$$

has f.f. <u>right</u> adjoint

cover by Gabriel localisations

<u>Roughly</u> $\mathcal{T}^{|\alpha|}$ is an α-perfect class of α-small objects (easy to show it generates $\therefore$ $\mathcal{T}$ is α-compactly gen and $\mathcal{T}^\alpha = \mathcal{T}^{|\alpha|}$)

$\Updownarrow$

π preserves coproducts $\quad (\operatorname{Ker} q_1 \cap \operatorname{Ker} q_2 = 0)$

$\Uparrow$

π_1, π_2 preserve coproducts

$\Uparrow$

$\mathcal{T}/\mathcal{I}_1, \mathcal{T}/\mathcal{I}_2$ are α-compactly generated.

3 Analogy (algebraic cases)

	Abelian	Triangulated
Basic example	$\mathrm{Mod}R$, R a ring $\mathrm{Mod}\,\mathcal{R}$, $\mathcal{R}$ small preadditive	$\mathbb{D}(R)$, R a ring $\mathbb{D}(\mathcal{C})$, $\mathcal{C}$ small DG-category
Fancy example	$\mathrm{Mod}R \xrightarrow{\pi} \mathcal{A}$, a localisation, i.e. π has a fully faithful right adjoint, $\mathcal{A} = \mathrm{Mod}R/\mathcal{B}$.	$\mathbb{D}(\mathcal{C}) \longrightarrow \mathcal{T}$ a localisation, i.e. there is localising $\mathcal{S} \subseteq \mathbb{D}(\mathcal{C})$, $\mathcal{S} = \langle \mathcal{S}_0 \rangle$, $\mathcal{S}_0$ a set, with $\mathcal{T} = \mathbb{D}(\mathcal{C})/\mathcal{S}$.
Gabriel-Popescu	$\mathcal{A}$ Grothendieck abelian $\Longrightarrow$ $\mathcal{A}$ is fancy	$\mathcal{T}$ well-generated and algebraic $\Longrightarrow$ $\mathcal{T}$ is fancy (Porta)
Filtrations	$\mathcal{A}$ Grothendieck $\Rightarrow$ $\mathcal{A} = \bigcup_\alpha \mathcal{A}^\alpha$	$\mathcal{T}$ well-generated $\Longrightarrow$ $\mathcal{T} = \bigcup_\alpha \mathcal{T}^\alpha$ α-compact
Adjoints	$F: \mathcal{A} \to \mathcal{B}$ colim pres. + $\mathcal{A}$ Grothendieck $\Longrightarrow$ F has a right adjoint	$F: \mathcal{T} \to \mathcal{S}$ triangulated and $\coprod$-pres, + $\mathcal{T}$ well-generated $\Longrightarrow$ F has a right adjoint.