Frobenius Algebras and Monoidal Categories.

(1) The Frobenius relation.
(2) The characterisation of Frobenius algebras.
(3) The category of Frobenius algebras.

Recap: Last time, we defined a Frobenius algebra as an associative algebra over $\mathbb{K}$ endowed with a nondegenerate form $\varepsilon : A \to \mathbb{K}$ called the Frobenius form. This gave rise to an associative nondegenerate pairing $\beta : A \otimes A \to \mathbb{K}$ with copairing γ given by

$$\gamma(1_{\mathbb{K}}) = \sum_i e_i \otimes \hat{e}_i$$

We then defined a comultiplication $\delta : A \to A \otimes A$ by

$$\delta = (\mu \otimes id) \circ (id \otimes \gamma) \quad , \quad \delta(a) = \sum_i a e_i \otimes \hat{e}_i .$$

making A a coalgebra with counit ε, so that

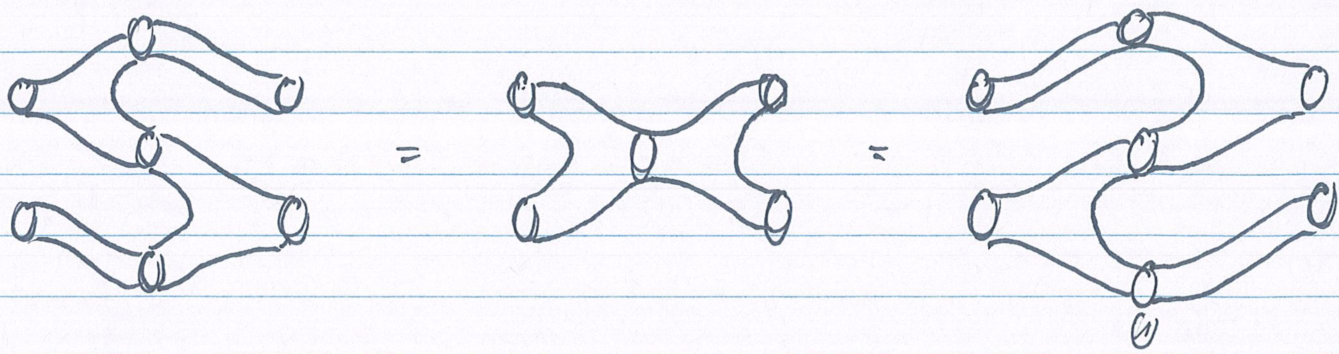
(1) $(\delta \otimes id)\delta = (id \otimes \delta)\delta$
(2) $(\varepsilon \otimes id)\delta = id = (id \times \varepsilon)\delta$.

<u>Lemma</u>: let A be a Frobenius algebra with multiplication μ and comultiplication δ as defined above. Then the following diagram commutes:

$$\begin{array}{ccccc} & & A\otimes A & & \\ & \swarrow^{\delta\otimes id} & \downarrow{\mu} & \searrow^{id\otimes\delta} & \\ A\otimes A\otimes A & & A & & A\otimes A\otimes A \\ & \searrow_{id\otimes\mu} & \downarrow{\delta} & \swarrow_{\mu\otimes id} & \\ & & A\otimes A & & \end{array}$$

We refer to this as the Frobenius relation.

This may look at first like an opaque and technical relation, but in fact it is quite natural when viewed in terms of cobordisms:



proof: let $a, b \in A$.

$$(\delta \circ \mu)(a \otimes b) = \delta(ab) = \sum_i ab\, e_i \otimes \hat{e}_i$$

$$(id \otimes \mu) \circ (\delta \otimes id)(a \otimes b) = \sum_i (id \otimes \mu)(a e_i \otimes \hat{e}_i \otimes b)$$

$$= \sum_i a e_i \otimes b \hat{e}_i$$

$$= \sum_i ab\, e_i \otimes \hat{e}_i$$

$$(\mu \otimes id) \circ (id \otimes \delta)(a \otimes b) = \sum_i (\mu \otimes id)(a \otimes b e_i \otimes \hat{e}_i)$$

$$= \sum_i ab\, e_i \otimes \hat{e}_i .$$

$\square$.

<u>Lemma</u>: the following relations hold.

(1) $\gamma = \delta \circ \eta$.
(2) $\eta = \varepsilon \circ \gamma$.

proof: $\gamma(1) = \sum_i e_i \otimes \hat{e}_i = \delta(1_A) = \delta(\eta(1))$

$$\eta(1) = 1_A = \sum_i \varepsilon(e_i \otimes \hat{e}_i) = \varepsilon(\gamma(1)) .$$

$\square$.

The reason for the name Frobenius condition is that it characterizes Frobenius algebras not only among associative algebras of finite dimension, but also among general vector spaces equipped with multiplication. The next theorem makes this specific:

<u>Theorem:</u> Let A be a vector space with a multiplication map $\mu : A \otimes A \to A$, with unit $\eta : \mathbb{K} \to A$, a comultiplication $\delta : A \to A \otimes A$, and with counit $\varepsilon : A \to \mathbb{K}$. Suppose that the Frobenius relation holds. Then

(1) A is of finite dimension.
(2) The multiplication μ is associative, and the comultiplication δ is coassociative.
(3) The counit ε is a Frobenius form, and thus (A, ε) is a Frobenius algebra.

proof: Define a pairing $\beta := \varepsilon\mu$ and copairing $\gamma := \delta\eta$. We will show that β is nondegenerate, and thus that A is finite dimensional. We must show that the compositions

$$A \xrightarrow{id \otimes \gamma} A \otimes A \otimes A \xrightarrow{\beta \otimes id} A$$

$$A \xrightarrow{\gamma \otimes id} A \otimes A \otimes A \xrightarrow{id \otimes \beta} A$$

are equal to the identity.

The Frobenius relation states that

$$(id\otimes\mu)\circ(\delta\otimes id) = \delta\circ\mu$$

so $(id\otimes\varepsilon)\circ(id\otimes\mu)\circ(\delta\otimes id)\circ(\eta\otimes id) = (id\otimes\varepsilon)\circ\delta\circ\mu\circ(\eta\otimes id)$.

By definition the left hand side is equal to

$$(id\otimes\beta)\circ(\gamma\otimes id)$$

Moreover, the unit and counit conditions on δ and μ show that the right hand side is equal to the identity. Hence, the composition

$$A \xrightarrow{\gamma\otimes id} A\otimes A\otimes A \xrightarrow{id\otimes\beta} A$$

is the identity. Applying the second part of the Frobenius relation gives the other required composition in the same way.

Next, we need to show that μ is associative and δ is coassociative, that is

(1) $\mu\circ(\mu\otimes id) = \mu\circ(id\otimes\mu)$

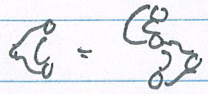
(2) $(\delta\otimes id)\circ\delta = (id\otimes\delta) = \delta$.

Once again, taking the Frobenius relation

$$(id \otimes \mu) \circ (\delta \otimes id) = \delta \circ \mu$$

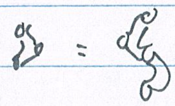
We can precompose with $\eta \otimes id$ to give

$$\delta = (id \otimes \mu) \circ (\gamma \otimes id)$$



Likewise, postcomposing with $id \otimes \varepsilon$ gives

$$\mu = (id \otimes \beta) \circ (\delta \otimes id).$$



only way we could define this.

Now we can write

$$\mu \circ (id \otimes \mu) = \mu \circ (id \otimes \beta \otimes id) \circ (id \otimes \delta \otimes id)$$

$$= (id \otimes \beta) \circ (\mu \otimes id) \circ (id \otimes \delta \otimes id)$$

Applying Frobenius

$$= (id \otimes \beta) \circ (\delta \otimes id) \circ (\mu \otimes id)$$

$$= \mu \circ (\mu \otimes id).$$

Likewise

$$(\delta \otimes id)\delta = (id \otimes \mu \otimes id) \circ (id \otimes \delta) \circ (\gamma \otimes id)$$

$$= (id \otimes \delta) \circ (id \otimes \mu) \circ (\gamma \otimes id)$$

$$= (id \otimes \delta) \circ \delta.$$

So μ is associative, and δ coassociative.

Finally, since μ is associative, the pairing $\beta = \varepsilon\mu$ is associative as well, so (A, β) is a Frobenius algebra.

□.

note: this allows us to define a "Frobenius object" in any Monoidal cat.

(3) The category of Frobenius algebras:

Recall that an algebra (resp. coalgebra) homomorphism is a linear map which commutes with multiplication (resp. comultiplication).

lemma: if a $\Bbbk$-algebra homomorphism ϕ between two Frobenius algebras (A, ε), (A, ε') is compatible with the forms in the sense that $\varepsilon = \varepsilon'\phi$, then ϕ is injective.

proof: If ϕ is compatible with the Frobenius forms, then the kernel of ϕ is contained in $Null(\varepsilon)$. But $Null(\varepsilon)$ contains no nontrivial ideals, so $Null(\varepsilon) = 0$ and ϕ is injective.

□.

A Frobenius algebra homomorphism $\phi:(A,\varepsilon)\to(A',\varepsilon')$ between two Frobenius algebras is an algebra homomorphism which is also a coalgebra homomorphism. In particular it preserves the Frobenius form so that $\varepsilon = \varepsilon'\phi$.

We denote by $FA_{\mathbb{K}}$ the category of Frobenius algebras over $\mathbb{K}$ and Frobenius algebra homomorphism.

Lemma: all morphisms in $FA_{\mathbb{K}}$ are isomorphisms.

proof: Write $A^* := Hom_{\mathbb{K}}(A, \mathbb{K})$. Then A'^* and A^* are both Frobenius algebras. Thus, the dual map $\phi^* : A'^* \to A^*$ is multiplicative and respects units and counits, and is thus compatible with the Frobenius forms. Thus, ϕ^* is injective. Since A is a finite dimensional vector space, this means ϕ is surjective. But ϕ is already injective, so it is invertible.

□.

THEOREM: The skeletal cobordism category $(\underline{2Cob}, \amalg, \emptyset, T)$ is the free symmetric monoidal category containing a (co)commutative Frobenius object.
That is, given a Frobenius object A in a symmetric monoidal category $(V, \otimes, \mathbb{I}, \tau)$, there is a unique symmetric monoidal functor

$$\underline{2cob} \to V \quad , \quad \mathbb{1} \mapsto A$$

giving a canonical equivalence of categories

$$SymModCat(\underline{2cob}, V) \simeq cFrob(V).$$