Frobenius Algebras

This talk aims to introduce three equivalent definitions of Frobenius algebras and to demonstrate that the Frobenius structure gives rise to a unique coalgebra structure.

Outline:

(1) Pairings of vector spaces.
(2) Frobenius algebras
(3) Examples
(4) Coalgebra construction from the Frobenius pairing.

(1) Bilinear Pairings:

Def: Let V and W be $\mathbb{K}$-vector spaces. A bilinear pairing, or just pairing, of V with W is a linear map

$$\beta : V \otimes W \longrightarrow \mathbb{K} \quad , \quad v \otimes w \longmapsto \langle v, w \rangle$$

We say that a pairing β is nondegenerate if the following hold:

1. $\forall w \ \langle v_0, w \rangle = 0 \Rightarrow v_0 = 0.$
2. $\forall v \ \langle v, w_0 \rangle = 0 \Rightarrow w_0 = 0.$

note: We can equivalently characterize a nondegenerate pairing $\beta : V \otimes W \longrightarrow \mathbb{K}$ as inducing linear injections

$$\begin{aligned} V &\longrightarrow W^* \\ v &\longmapsto \langle v, \_ \rangle \end{aligned} \quad , \quad \begin{aligned} W &\longrightarrow V^* \\ w &\longmapsto \langle \_, w \rangle \end{aligned} .$$

remark: a third equivalent definition of nondegeneracy can be given as the existence of a linear map $\gamma : \mathbb{K} \longrightarrow W \otimes V$, called a copairing, such that the following composites are equal to the identity maps of V and W respectively:

$$V \cong V \otimes \mathbb{K} \xrightarrow{id \otimes \gamma} V \otimes W \otimes V \xrightarrow{\beta \otimes id} \mathbb{K} \otimes V \cong V$$

$$W \cong \mathbb{K} \otimes W \xrightarrow{\gamma \otimes id} W \otimes V \otimes W \xrightarrow{id \otimes \beta} W \otimes \mathbb{K} \cong W$$

lemma 1: if the pairing $\beta : V \otimes W \to \mathbb{K}$ is nondegenerate then W, V are finite dimensional.

proof: Suppose β is nondegenerate with copairing γ, and suppose $1_{\mathbb{K}} \mapsto \sum_{i=1}^{n} w_i \otimes v_i$ under γ, with $v_i \in V$ $w_i \in W$. Take $x \in W$ and send it through the composite

$$\begin{aligned} W &\longrightarrow W \otimes V \otimes W \longrightarrow W \\ x &\mapsto \sum_{i=1}^{n} w_i \otimes v_i \otimes x \longmapsto \sum_{i=1}^{n} w_i \langle v_i, x \rangle \end{aligned}$$

Nondegeneracy means that this is the identity map, so $x = \sum_{i=1}^{n} w_i \langle v_i, x \rangle$, and hence $W = \text{span}\{w_1, \dots, w_n\}$ is finite dimensional.
The same holds for V.

$\square$.

<u>Example</u>: the evaluation pairing $V \otimes V^* \to \mathbb{K}$, $v \otimes \lambda \mapsto \lambda(v)$ is nondegenerate if and only if V is of finite dimension.

(2) <u>Frobenius Algebras</u>:

<u>Def</u>: a $\mathbb{K}$-algebra is a $\mathbb{K}$-vector space A together with two $\mathbb{K}$-linear maps

$$\mu : A \otimes A \longrightarrow \mathbb{K} \qquad \eta : \mathbb{K} \longrightarrow A$$

called multiplication and unit respectively, such that the following commute:

$$\begin{tikzcd}
 & A \otimes A \otimes A \arrow[dl, "\mu \otimes id"'] \arrow[dr, "id \otimes \mu"] & \\
A \otimes A \arrow[dr, "\mu"'] & & A \otimes A \arrow[dl, "\mu"] \\
 & A &
\end{tikzcd}$$

(associativity)

$$\begin{tikzcd}
A \otimes \mathbb{K} \arrow[r, "\eta \otimes id"] \arrow[dr, "l"'] & A \otimes A \arrow[d, "\mu"] \\
 & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \otimes A \arrow[d, "\mu"'] & \mathbb{K} \otimes A \arrow[l, "id \otimes \eta"'] \arrow[dl, "r"] \\
A &
\end{tikzcd}$$

(unity)

note: a $\mathbb{K}$-algebra is a ring A equipped with a ring homomorphism $\mathbb{K} \to A$.

<u>Def</u>: A <u>Frobenius algebra</u> is a finite dimensional $\mathbb{K}$-algebra A, equipped with a linear functional $\varepsilon : A \to \mathbb{K}$ whose nullspace contains no nontrivial left ideals. The functional $\varepsilon \in A^*$ is called a <u>Frobenius form</u>.

There are actually at least two other definitions of Frobenius algebras:

<u>Def</u>: A Frobenius algebra is a finite dimensional $\mathbb{K}$-algebra A, equipped with an <u>associative</u> nondegenerate pairing $\beta : A \otimes A \to \mathbb{K}$ called the <u>Frobenius pairing</u>.

<u>Def</u>: A Frobenius algebra is a finite dimensional $\mathbb{K}$-algebra A, equipped with a left A-isomorphism $\theta : A \xrightarrow{\sim} A^*$.

<u>Proposition</u>: these three definitions of Frobenius algebras are equivalent.

proof: (1) ⇒ (2) given a linear functional $\varepsilon \in A^*$ with no nontrivial left ideals, define a pairing $\beta : A \otimes A \to \mathbb{K}$ by

$$x \otimes y \mapsto \varepsilon(xy).$$

This is clearly associative, and $\varepsilon(Ay) = 0 \Rightarrow y = 0$ implies $\forall x \langle x, y \rangle = 0 \Rightarrow y = 0$, so β is nondegenerate.

proof: (cont.) (2) ⇒ (1): given an associative nondegenerate pairing $\beta: A \otimes A \to \mathbb{K}$, define a linear functional $\varepsilon: A \to \mathbb{K}$ by

$$\varepsilon(x) := \langle x, 1_A \rangle = \langle 1_A, x \rangle.$$

Then $\varepsilon(Ax) = 0$ implies $\langle Ax, 1_A \rangle = \langle x, A \rangle = 0$, so $x = 0$ since β is nondegenerate, so ε has no non-trivial left ideals.

(2) ⇒ (3): Let $\beta: A \otimes A \to \mathbb{K}$ be a nondegenerate pairing. Define $\theta: A \to A^*$ by

$$\theta(y)(x) := \langle x, y \rangle.$$

Then the nondegeneracy of β makes θ injective, and hence an isomorphism since A is finite dimensional.

(3) ⇒ (2): converse to the above. Given an isomorphism $\theta: A \to A^*$, define a pairing $\beta: A \otimes A \to \mathbb{K}$ by

$$\langle x, y \rangle := \theta(y)(x)$$

Then β is nondegenerate since θ is an isomorphism: $\langle -, y \rangle = 0 \Rightarrow \theta(y) = 0 \Rightarrow y = 0$. $\langle x, - \rangle = 0 \Rightarrow x = 0$.
Associativity follows from

$$\langle xy, z \rangle = \theta(xy)(z) = \theta(x)(yz) = \langle x, yz \rangle$$

$\square$.

remark: a consequence of the third definition is that given a Frobenius form ε, every other Frobenius form on A is given by precomposing ε with multiplication by a unit in A.

Def: A Frobenius algebra A is called symmetric if the Frobenius pairing on A is symmetric.

(3) Examples:

(a) The complex numbers $\mathbb{C}$ as an algebra over $\mathbb{R}$ is a Frobenius algebra with Frobenius form

$$\begin{aligned} \mathrm{Re} : \mathbb{C} &\longrightarrow \mathbb{R} \\ a + bi &\longmapsto a \end{aligned}$$

(b) The set of $n \times n$ matrices $\mathrm{Mat}_n(\mathbb{K})$ over $\mathbb{K}$ is a Frobenius algebra with Frobenius form

$$\begin{aligned} \mathrm{Tr} : \mathrm{Mat}_n(\mathbb{K}) &\longrightarrow \mathbb{K} \\ M &\longmapsto \mathrm{Tr}(M) \end{aligned}$$

the trace map. This is an example of a symmetric Frobenius algebra.

(c) Let $G = \{t_0, \dots, t_n\}$ be a finite group written multiplicatively with $t_0 = 1$. We define the group algebra

$$\mathbb{K}G = \{ \sum_i c_i t_i \mid c_i \in \mathbb{K} \}$$

with multiplication inherited from G. The algebra $\mathbb{K}G$ is given Frobenius structure by defining the Frobenius form

$$\begin{aligned} \varepsilon : \mathbb{K}G &\longrightarrow \mathbb{K} \\ t_i &\longmapsto \delta_{i,1} \end{aligned}$$

(d) Gorenstein rings.

(4) Coalgebra construction from the Frobenius pairing.

Def: A **coalgebra** over $\mathbb{K}$ is a vector space A together with two $\mathbb{K}$-linear maps

$$\delta : A \to A \otimes A \qquad \varepsilon : A \to \mathbb{K}.$$

called comultiplication and counit respectively, such that the following commute:

$$\begin{tikzcd}
 & A \otimes A \otimes A & \\
A \otimes A \arrow[ur, "\delta \otimes id"] & & A \otimes A \arrow[ul, "id \otimes \delta"'] \\
 & A \arrow[ul, "\delta"] \arrow[ur, "\delta"'] &
\end{tikzcd}$$

(coassociativity)

$$\begin{tikzcd}
\mathbb{K} \otimes A & A \otimes A \arrow[l, "\varepsilon \otimes id"'] & A \otimes A \arrow[r, "id \otimes \varepsilon"] & A \otimes \mathbb{K} \\
 & A \arrow[ul] \arrow[u, "\delta"'] & A \arrow[u, "\delta"] \arrow[ur] &
\end{tikzcd}$$

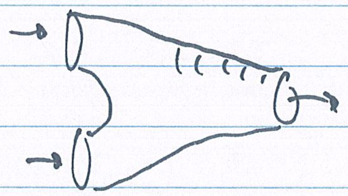
(counity)

<u>lemma</u>: Let A be a Frobenius algebra with a Frobenius pairing $\beta : A \otimes A \longrightarrow \mathbb{K}$. Then β induces a unique coalgebra structure on A with comultiplication $\delta : A \to A \otimes A$ given by

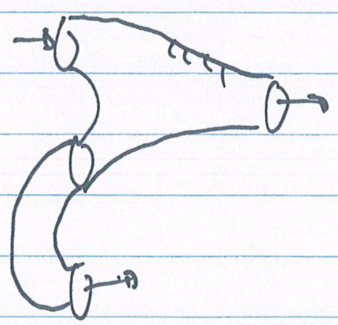
$$\delta = (m \otimes 1\!\!1_A) \circ (id \otimes \gamma)$$

where m is the multiplication on A, and γ is the copairing associated with β, and counit given by the Frobenius form $\varepsilon : A \to \mathbb{K}$.

note: this definition is motivated by the cobordism diagrams encountered in the first lecture. If we represent multiplication by a "pair of pants" with two inputs and one output,



it makes sense to attempt to construct a comultiplication by attaching another cobordism to create a diagram with one input and two outputs



proof: Since A is a Frobenius algebra, write $\theta : A \xrightarrow{\sim} A^*$ for the induced $\mathbb{k}$-isomorphism. Denote by $e_1, \dots, e_n$ a basis of A, and define $e_i^* \in A^*$

$$e_i^* \left(\sum_i a_i e_i \right) = a_i$$

Finally, write $\hat{e}_i = \theta^{-1}(e_i^*)$. Note that in general $\hat{e}_i \neq e_i$.

We can write our copairing explicitly as

$$1 \mapsto \sum_i e_i \otimes \hat{e}_i$$

To see that this is indeed the desired copairing, look at the composition

$$A \xrightarrow{1 \otimes \gamma} A \otimes A \otimes A \xrightarrow{\beta \otimes 1} A$$

$$a \mapsto \sum_i a \otimes e_i \otimes \hat{e}_i \longmapsto \sum_i \langle a, e_i \rangle \hat{e}_i$$

$$\theta(a) = \langle a, - \rangle = \theta \left(\sum_i \langle a, e_i \rangle \hat{e}_i \right)$$

So $a = \sum_i \langle a, e_i \rangle \hat{e}_i$. We will need this later.

To show:

(a) $(id \otimes \delta) \circ \delta = (\delta \otimes id) \circ \delta$ (coassociativity)
(b) $(\varepsilon \otimes id) \circ \delta = id = (id \otimes \varepsilon) \circ \delta$ (counity).

(a) $(id \otimes \delta) \circ \delta(a) = \sum_i^n (1 \otimes \delta)(ae_i \otimes \hat{e}_i)$

$$= \sum_i^n ae_i \otimes \delta(\hat{e}_i)$$

$$= \sum_{ij}^n ae_i \otimes \hat{e}_i e_j \otimes \hat{e}_j$$

$$(\delta \otimes id) \circ \delta(a) = \sum_i^n (\delta \otimes id)(ae_i \otimes \hat{e}_i)$$

$$= \sum_i^n \delta(ae_i) \otimes \hat{e}_i$$

$$= \sum_{ij}^n (ae_i)e_j \otimes \hat{e}_j \otimes \hat{e}_i$$

Now, $\hat{e}_1, \ldots, \hat{e}_n$ is a basis of A, so we can write

$$\hat{e}_i e_j = \sum_k \mu_k^{ij} \hat{e}_k \quad ; \quad e_i e_j = \sum_k \lambda_k^{ij} e_k .$$

Take the first, and apply θ to both sides:

$$\langle \hat{e}_i e_j, e_t \rangle = \sum_k \mu_k^{ij} \langle \hat{e}_k, e_t \rangle$$

$$\Rightarrow \langle \hat{e}_i, e_j e_t \rangle = \sum_k \mu_k^{ij} e_k^*(e_t)$$

$$\Rightarrow e_i^*(e_j e_t) = \mu_t^{ij}$$

But $e_i^*(e_j e_t) = \lambda_i^{jt}$

So $\mu_t^{ij} = \lambda_i^{jt}$

Thus,

$$(\delta \otimes id) \circ \delta(a) = \sum_{ijt} a \lambda_t^{ij} e_t \otimes \hat{e}_j \otimes \hat{e}_i$$

$$(id \otimes \delta) \circ \delta(a) = \sum_{ijt} a e_i \otimes \mu_t^{ij} \hat{e}_t \otimes \hat{e}_j$$

$$= \sum_{ijt} a \lambda_i^{jt} e_i \otimes \hat{e}_t \otimes \hat{e}_j$$

So after rearranging indices we have

$$(\delta \otimes id) \circ \delta = (id \otimes \delta) \circ \delta.$$

b. $(\varepsilon \otimes id) \circ \delta(a) = \sum_i (\varepsilon \otimes id) a e_i \otimes \hat{e}_i$

$$= \sum_i \langle a, e_i \rangle \hat{e}_i$$

$$= a$$, from the copairing relation.

The other follows from the copairing relation in the same way.

$\square$.

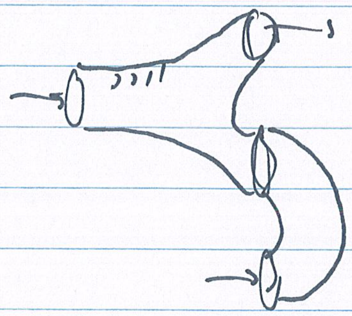
Thus, $(\delta\otimes id)\circ\delta(a) = \sum_{ijt} a\lambda_t^{ij} e_t \otimes \hat{e}_j \otimes \hat{e}_i$

$$(id\otimes\delta)\circ\delta(a) = \sum_{ijt} a e_i \otimes \mu_t^{ij}\hat{e}_t \otimes \hat{e}_j$$

$$= \sum_{ijt} a\lambda_i^{jt} e_i \otimes \hat{e}_t \otimes e_j$$

$$\therefore (\delta\otimes id)\circ\delta(a) = (id\otimes\delta)\circ\delta(a).$$

As for uniqueness, we can again appeal to graphical calculus to construct multiplication out of the comultiplication and pairing:



So we write $m = (id\circ\beta)\circ(\delta\circ id)$. Although we will not go into details, it is the uniqueness of the copairing - pairing pair which makes this comultiplication - multiplication pair unique, by construction.

$\square$.