Constructing A_∞-categories of matrix factorisations

Daniel Murfet

11/7/19

therisingsea.org

- Computation is the process of transforming information from an implicit form to an explicit form (formally: cut-elimination in sequent calculus).

- Aim compute the DG-category of matrix factorisations of a potential W.

References

- T. Dyckerhoff, D.M., "Pushing forward matrix factorisations" Duke. J. 2013

- D.M., "The cut operation on matrix factorisations" JPAA 2018.

- D.M., "Constructing A_∞-categories of matrix factorisations" arXiv: 1903.07211.
  (see also therisingsea.org for working notes).

Preliminaries

Potentials Let k be a commutative $\mathbb{Q}$-algebra, then $W \in R = k[x_1, \dots, x_n]$ is called a <u>potential</u> if

(i) $\partial_{x_1} W, \dots, \partial_{x_n} W$ is quasi-regular

(ii) $R/(\partial_{x_1} W, \dots, \partial_{x_n} W)$ is a f.g. free k-module

(iii) the Koszul complex of $\partial_{x_1} W, \dots, \partial_{x_n} W$ is exact outside deg. 0.

Defn The DG-category $\mathcal{A} = mf(R, W)$ has

- <u>objects</u> f. rank matrix factorisations of W, i.e. X & $d_X^2 = W \cdot 1_X$.
- <u>morphisms</u> $\mathcal{A}(X, Y) = \left(\mathrm{Hom}_R(X, Y),\ \alpha \mapsto d_Y \alpha - (-1)^{|\alpha|} \alpha d_X \right)$.

This is a $\mathbb{Z}_2$-graded DG-category over R.

Preliminaries

Def^n A small $\mathbb{Z}_2$-graded A_∞-category $\mathcal{B}$ over k has a set $ob(\mathcal{B})$ of objects, and $\mathbb{Z}_2$-graded k-modules $\mathcal{B}(a,b)$ for all $a,b \in ob(\mathcal{B})$ equipped with suspended forward compositions which are odd linear maps

$$r_{a_0,\dots,a_n} : \mathcal{B}(a_0,a_1)[1] \otimes \cdots \otimes \mathcal{B}(a_{n-1},a_n)[1] \longrightarrow \mathcal{B}(a_0,a_n)[1]$$

r_n

satisfying the A_∞-constraints (without explicit signs)

$$\sum_{\substack{i \geq 0,\, j \geq 1 \\ 1 \leq i+j \leq n}} r_{a_0,\dots,a_i,a_{i+j},\dots,a_n} \circ \left(id_{a_0a_1} \otimes \cdots \otimes r_{a_i,\dots,a_{i+j}} \otimes \cdots \otimes id_{a_{n-1},a_n} \right) = 0$$

Example Any $\mathbb{Z}_2$-graded DG-category, $r_n = 0$ for $n \geq 3$.

Finite A_∞-model

Let $\varphi : k \longrightarrow R$ be a morphism of commutative rings, $\mathcal{A}$ a DG-category over R.

Restriction of scalars gives a functor

$$\begin{tikzcd} A_\infty\text{-cat}(R) \arrow[d, "\varphi_*"] & \mathcal{A} \arrow[d, mapsto] & \\ A_\infty\text{-cat}(k) & \varphi_*(\mathcal{A}) \arrow[r, shift left, "F"] & \mathcal{B} \arrow[l, shift left, "G"] \end{tikzcd}$$

may have $r_1 \neq 0$

<u>Def^n</u> A <u>finite A_∞-model</u> of $\mathcal{A}$ over k is an A_∞-category $\mathcal{B}$ over k with all Hom-spaces f.g. projective$/k$, A_∞-functors F, G and A_∞-homotopies $F \circ G \overset{\infty}{\simeq} 1$, $G \circ F \overset{\infty}{\simeq} 1$.

Minimal A_∞-model

Let $\varphi : k \longrightarrow R$ be a morphism of commutative rings, $\mathcal{A}$ a DG-category over R.

Restriction of scalars gives a functor

$$\begin{tikzcd} A_\infty\text{-cat}(R) \arrow[d, "\varphi_*"] & & \mathcal{A} \arrow[d, mapsto] & \\ A_\infty\text{-cat}(k) & & \varphi_*(\mathcal{A}) \arrow[r, shift left, "F"] & (H^*(\mathcal{A}), \{r_n\}_{n\geq 2}) \arrow[l, shift left, "G"] \end{tikzcd}$$

<u>Def^n</u> A <u>minimal A_∞-model</u> of $\mathcal{A}$ over k is an A_∞-structure $\{r_n\}_{n\geq 1}$ on $H^*(\mathcal{A})$ with $r_1 = 0$, r_2 induced by composition, and A_∞-functors F, G and A_∞-homotopies $F\circ G \overset{\infty}{\simeq} 1$, $G \circ F \overset{\infty}{\simeq} 1$.

c.f. Remark 1.13 Seidel's book on Fukaya categories.

Idempotent finite A_∞-models

Let $\varphi : k \longrightarrow R$ be a morphism of commutative rings, $\mathcal{A}$ a DG-category over R.

Restriction of scalars gives a functor

$$
\begin{tikzcd}
A_\infty\text{-cat}(R) \arrow[d, "\varphi_*"] \\
A_\infty\text{-cat}(k)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{A} \arrow[d, mapsto] & & \\
\varphi_*(\mathcal{A}) \arrow[rr, shift left, "F"] & & \mathcal{B} \arrow[ll, shift left, "G"] \arrow[loop right, "E"]
\end{tikzcd}
$$

may have $r_1 \neq 0$.

Def^n An <u>idempotent finite A_∞-model</u> of $\mathcal{A}$ over k is an A_∞-category $\mathcal{B}$ with all Hom-spaces f.g. projective$/k$, A_∞-functors F, G, E as above and A_∞-homotopies $F \circ G \overset{\infty}{\simeq} E$, $G \circ F \overset{\infty}{\simeq} 1$. ($E = 1$ gives finite models)

Why finite models?

idempotent finite model	finite model	minimal model
$(\mathcal{B}, E_1, E_2, \dots, r_1, r_2, r_3, \dots)$	$(\mathcal{B}, r_1, r_2, r_3, \dots)$	$(H^*(\mathcal{A}), r_2, r_3, \dots)$
12 $A_\infty\text{-cat}(k)^\omega$	12	12
$(\mathcal{A}, 1, r_1, r_2)$	$(\mathcal{A}, r_1, r_2)$	$(\mathcal{A}, r_1, r_2)$

- String field theory (A_∞) vs. topological field theory (Δed).
  $(H^*(A), r_2, r_3, \dots)$ $\qquad$ $(H^*(A), r_2)$

- The information in higher products is important (e.g. for studying moduli).

The question is: which kind of finite model best packages this information?

Physics refs. Lazaroiu (JHEP 2001), Lazaroiu-Roiban (JHEP 2002), Lazaroiu (2006), Carqueville-Dowdy-Recknagel (JHEP 2012), Carqueville-Kay (CMP 2012), Baumgartl-Brunner-Gaberdiel (JHEP 2007), Baumgartl-Wood (JHEP 2009), Knapp-Omer (JHEP 2006).

Melbourne

<u>idempotent finite model</u>

$$(\mathcal{B}, E_1, E_2, \dots, r_1, r_2, r_3, \dots)$$

|2

$$(\mathcal{A}, 1, r_1, r_2)$$

<u>minimal model</u>

$$(H^*(\mathcal{A}), r_2, r_3, \dots)$$

|2

$$(\mathcal{A}, r_1, r_2)$$

k a field

<u>idempotent finite model</u>

$(\mathcal{B}, E_1, E_2, \dots, r_1, r_2, r_3, \dots)$

12

$(\mathcal{A}, 1, r_1, r_2)$

<u>minimal model</u>

$(H^*(\mathcal{A}), r_2, r_3, \dots)$

12

$(\mathcal{A}, r_1, r_2)$

Choose k-linear homotopy equivalences

$$\mathcal{A}(a,b) \underset{g}{\overset{f}{\rightleftarrows}} H^*\mathcal{A}(a,b)$$

$gf = 1 - [d_{\mathcal{A}}, H]$ $\quad fg = 1$

and transfer A_∞-structure to $H^*(\mathcal{A})$

- useful for special objects (e.g. k^{stab})
  (Seidel, Dyckerhoff, Efimov, Sheridan)
- depends on k being a field.

References

- P. Seidel, "Homological mirror symmetry for the genus two curve" arXiv: 0812.1171.
- T. Dyckerhoff, "Compact generators in categories of matrix factorisations" Duke Math. J. 2011.
- A. Efimov, "Homological mirror symmetry for curves of higher genus" Adv. Math. 2012.
- N. Sheridan, "Homological mirror symmetry for Calabi-Yau hypersurfaces in projective space" Inventiones 2015.
- D. Shklyarov, "Calabi-Yau structures on categories of matrix factorisations" J. of Geometry and Physics 2017.
- J. Tu, "Categorical Saito theory I: a comparison result" arXiv: 1902.04596.

Prove $J \cong J'$ by finding generators G, G' and A_∞-iso $\mathrm{End}(G) \cong \mathrm{End}(G')$.

<u>idempotent finite model</u>

$$(\mathcal{B}, E_1, E_2, \ldots, r_1, r_2, r_3, \ldots)$$

12

$$(A, 1, r_1, r_2)$$

- Exists for all of $A = mf(W)$
- Constructive when Gröbner methods are available (e.g. k a field or poly. ring).
- Downside: not minimal. However, we know TFT formulas (HRR, Kapustin-Li) can be derived directly from $\mathcal{B}$, E_1.
- For special objects can split E.
- <u>Key point</u>: first enlarge A!

<u>minimal model</u>

$$(H^*(A), r_2, r_3, \ldots)$$

12

$$(A, r_1, r_2)$$

Choose k-linear homotopy equivalences

$$A(a,b) \underset{g}{\overset{f}{\rightleftarrows}} H^*A(a,b)$$

and transfer A_∞-structure to $H^*(A)$

- useful for special objects (e.g. k^{stab}) (Seidel, Dyckerhoff, Efimov, Sheridan, Tu)
- depends on k being a field.

idempotent finite model

$$(\mathcal{B}, E_1, E_2, \ldots, r_1, r_2, r_3, \ldots)$$

12

$$(\mathcal{A}, 1, r_1, r_2)$$

- $W \in R = k[x_1, \ldots, x_n]$ a potential
- $\mathcal{A} = \mathrm{mf}(R, W)$.
- $I = (t_1, \ldots, t_n)$, $t_i := \partial_{x_i} W$.
- Given MFs X, Y

f.g. proj / k

$$\mathcal{B}(X, Y) = R/I \otimes_R \mathrm{Hom}_R(X, Y)$$

idempotent finite model

$(\mathcal{B}, E_1, E_2, \ldots, r_1, r_2, r_3, \ldots)$

12

$(A, 1, r_1, r_2)$

- $W \in R = k[x_1, \ldots, x_n]$ a potential
- $\mathcal{A} = mf(R, W)$.
- $I = (t_1, \ldots, t_n)$, $t_i := \partial_{x_i} W$.
- Given MFs X, Y

f.g. proj/k

$$\mathcal{B}(X,Y) = R/I \otimes_R \mathrm{Hom}_R(X,Y)$$

Theorem (M) There exists an idempotent finite A_∞-model of $\mathcal{A}$ as above, with r_1, r_2 on $\mathcal{B}$ induced from $\mathcal{A}$, and

$$E_1 \simeq \gamma_n \cdots \gamma_1 \gamma_1^{\dagger} \cdots \gamma_n^{\dagger}$$

with $\gamma_i = At_i = [d_{\mathcal{A}}, \frac{\partial}{\partial t_i}]$ and, for homotopies $[\lambda_i, d_{\mathcal{A}}] = t_i \cdot 1_{\mathcal{A}}$,

$$\gamma_i^{\dagger} = -\lambda_i - \sum_{m \geq 1} \sum_{q_1, \ldots, q_m} \frac{1}{(m+1)!} [\lambda_{q_m}, [\cdots [\lambda_{q_1}, \lambda_i] \cdots] At_{q_1} \cdots At_{q_m}$$

Connections and Residues

⌜following Lipman⌟

Let k be a commutative $\mathbb{Q}$-algebra, R a k-algebra, and $t_1, \dots, t_n$ a quasi-regular sequence in R such that R/I is f.g. projective over k, $I = (t_1, \dots, t_n)$.

Lemma (Formal tubular neighbourhood) Any k-linear section σ of $R \longrightarrow R/I$ induces an isomorphism of $k[[t_1, \dots, t_n]]$-modules

$$\sigma^* : R/I \otimes_k k[[t_1, \dots, t_n]] \longrightarrow \hat{R}$$

I-adic

not an iso of <u>algebras</u>
unless R/I is smooth.

Connections and Residues

Let k be a commutative $\mathbb{Q}$-algebra, R a k-algebra, and $t_1, \dots, t_n$ a quasi-regular sequence in R such that R/I is f.g. projective over k, $I = (t_1, \dots, t_n)$.

Lemma (Formal tubular neighbourhood) *Any k-linear section σ of $R \longrightarrow R/I$ induces an isomorphism of $k[[t_1, \dots, t_n]]$-modules*

$$\sigma^* : R/I \otimes_k k[[t_1, \dots, t_n]] \longrightarrow \hat{R}$$

defined by

$$(\sigma^*)^{-1}(r) = \sum_{M \in \mathbb{N}^n} r_M \otimes t^M$$

where the $r_M \in R/I$ are unique such that in $\hat{R}$ we have

$$r = \sum_{M \in \mathbb{N}^n} \sigma(r_M) \, t^M.$$

Connections and Residues

Let k be a commutative $\mathbb{Q}$-algebra, R a k-algebra, and $t_1, \ldots, t_n$ a quasi-regular sequence in R such that R/I is f.g. projective over k, $I = (t_1, \ldots, t_n)$.

<u>Lemma</u> (Formal tubular neighbourhood) Any k-linear section σ of $R \longrightarrow R/I$ induces an isomorphism of $k[[t_1, \ldots, t_n]]$-modules

Q: where does this come from?

$$\sigma^* : R/I \otimes_k k[[t_1, \ldots, t_n]] \longrightarrow \hat{R}$$

defined by

$$(\sigma^*)^{-1}(r) = \sum_{M \in \mathbb{N}^n} r_M \otimes t^M$$

where the $r_M \in R/I$ are unique such that in $\hat{R}$ we have

$$r = \sum_{M \in \mathbb{N}^n} \sigma(r_M) t^M.$$

Q: can we actually compute these?

[e.g. $R = k[x_1, \ldots, x_n]$, W a potential and $\underline{t} = (\partial_{x_1} W, \ldots, \partial_{x_n} W)$]

Connections and Residues

Let k be a commutative $\mathbb{Q}$-algebra, R a k-algebra, and $t_1, \ldots, t_n$ a quasi-regular sequence in R such that R/I is f.g. projective over k, $I = (t_1, \ldots, t_n)$.

$$\sigma^* : R/I \otimes_k k[[t_1, \ldots, t_n]] \xrightarrow{\cong} \hat{R}$$

If k is a field, $R = k[x_1, \ldots, x_n]$ Buchberger's algorithm produces a <u>Gröbner basis</u> $G = (g_1, \ldots, g_c)$ for I and expressions $g_i = \sum_{j=1}^{c} h_{ij} t_j \quad 1 \le i \le c.$

<u>Theorem</u> (Generalised Euclidean division) Given $r \in R$ we have

$$r = h + \bar{r}^G \longleftarrow \text{remainder upon division by } G.$$

for a unique pair $h, \bar{r}^G$ with $h \in I$ and no term of $\bar{r}^G$ divisible by the leading term of any g_i, $1 \le i \le c$.

Connections and Residues

Let k be a commutative $\mathbb{Q}$-algebra, R a k-algebra, and $t_1, \dots, t_n$ a quasi-regular sequence in R such that R/I is f.g. projective over k, $I = (t_1, \dots, t_n)$.

$$\sigma^* : R/I \otimes_k k[[t_1, \dots, t_n]] \xrightarrow{\cong} \hat{R}$$

If k is a field, $R = k[x_1, \dots, x_n]$, Gröbner basis $G = (g_1, \dots, g_c)$ for I
$g_i = \sum_j h_{ij} t_j$, remainders denoted $\overline{r}^G$.

<u>Lemma</u> $\sigma : R/I \longrightarrow R$, $\sigma(r) = \overline{r}^G$ is a k-linear section of $R \longrightarrow R/I$

Connections and Residues

Let k be a commutative $\mathbb{Q}$-algebra, R a k-algebra, and $t_1, \dots, t_n$ a quasi-regular sequence in R such that R/I is f.g. projective over k, $I = (t_1, \dots, t_n)$.

$$\sigma^* : R/I \otimes_k k[[t_1, \dots, t_n]] \xrightarrow{\cong} \hat{R}$$

If k is a field, $R = k[x_1, \dots, x_n]$, Gröbner basis $G = (g_1, \dots, g_c)$ for I
$g_i = \sum_j h_{ij} t_j$, remainders denoted $\overline{r}^G$.

<u>Lemma</u> $\sigma : R/I \longrightarrow R$, $\sigma(r) = \overline{r}^G$ is a k-linear section of $R \longrightarrow R/I$

$$\begin{aligned} r &= \overline{r}^G + h = \overline{r}^G + \textstyle\sum_{i=1}^c b_i g_i \\ &= \overline{r}^G + \textstyle\sum_{j=1}^c \left(\sum_{i=1}^c b_i h_{ij} \right) t_j \\ &= \overline{r}^G + \textstyle\sum_{j=1}^c \left(\sum_{i=1}^c \overline{b_i h_{ij}}^G + \sum_{i=1}^c u_{ij} g_i \right) t_j \\ &= \overline{r}^G + \textstyle\sum_{j=1}^c \left(\sum_{i=1}^c \overline{b_i h_{ij}}^G \right) t_j + \sum_{j,k=1}^c \left(\sum_{i=1}^c \overline{u_{ij} h_{ik}}^G \right) t_j t_k + \cdots \end{aligned}$$

r_M : $\quad r_0 \quad r_{(0,\dots,\underset{j}{1},\dots,0)} \quad r_{(0,\dots,\underset{j}{1},\dots,\underset{k}{1},\dots,0)}$.

Connections and Residues

Let k be a commutative $\mathbb{Q}$-algebra, R a k-algebra, and $t_1, \dots, t_n$ a quasi-regular sequence in R such that R/I is f.g. projective over k, $I = (t_1, \dots, t_n)$.

Upshot If k is a field, $R = k[x_1, \dots, x_n]$, σ^* may be computed by Gröbner methods.

In general,

$$\sigma^* : R/I \otimes_k k[[t_1, \dots, t_n]] \xrightarrow{\cong} \hat{R} \qquad (k[[\underline{t}]]\text{-linear})$$

$$\sum_{M \in \mathbb{N}^n} \sigma(r_M)\, t^M = r$$

There is a k-linear connection $\nabla : \hat{R} \longrightarrow \hat{R} \otimes_{k[\underline{t}]} \Omega^1_{k[\underline{t}]/k}$, and

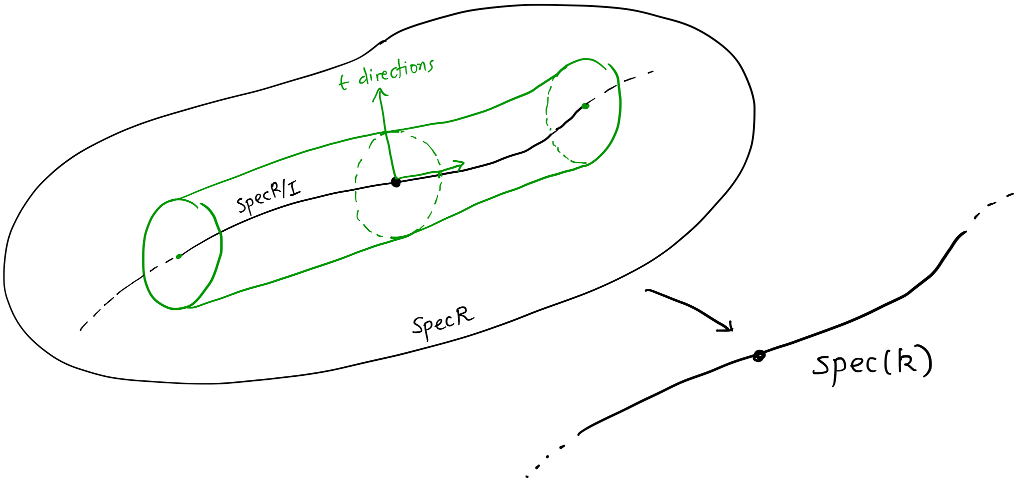
Theorem (Lipman, M) $\operatorname{Res}_{R/k}\begin{bmatrix} r\, dr_1 \cdots dr_n \\ t_1, \dots, t_n \end{bmatrix} = \operatorname{tr}_{R/I}\left(r [\nabla, r_1] \cdots [\nabla, r_n] \right)$

Connections and Residues

$$\sigma^* : R/I \otimes_k k[[t_1, \dots, t_n]] \xrightarrow{\cong} \hat{R}$$

$$\nabla : \hat{R} \longrightarrow \hat{R} \otimes_{k[t]} \Omega^1_{k[t]/k}$$

$$\mathrm{Res}_{R/k}\left[\frac{r\,dr_1 \cdots dr_n}{t_1, \dots, t_n}\right] = \mathrm{tr}_{R/I}\left(r[\nabla, r_1] \cdots [\nabla, r_n] \right)$$



Refining the Frobenius structure on Jac_W

$$\sigma^* : R/I \otimes_k k[[t_1,\dots,t_n]] \xrightarrow{\cong} \hat{R}$$

$$R/I \otimes_k R/I \xrightarrow{\sigma \otimes \sigma} \hat{R} \otimes \hat{R} \xrightarrow{\text{mult.}} \hat{R} \xrightarrow[\cong]{(\sigma^*)^{-1}} R/I \otimes k[[t_1,\dots,t_n]]$$

Choosing a k-basis $R/I = kz_1 \oplus \cdots \oplus kz_\mu$ this map computes

$$\sigma(z_i)\sigma(z_j) = \sum_{k=1}^{\mu} \sum_{\delta \in \mathbb{N}^n} T^{ij}_{k\delta}\, \sigma(z_k)\, t^\delta$$

The tensor T is one of the building blocks of the idempotent finite A_∞-model.

Example $W \in R = k[x_1,\dots,x_n]$ a potential, $t_i = \partial_{x_i} W$, $R/I = \mathrm{Jac}_W$.

The tensor T^{ij}_{k0} is the structural tensor of Jac_W as a Frobenius algebra.

An idempotent finite A_∞-model of mf

DG $\quad A = \mathrm{mf}(R, W) \qquad R = k[x_1, \dots, x_n] \qquad t_1 = \partial_{x_1} W, \dots, t_n = \partial_{x_n} W$

DG $\quad A_\theta = \bigwedge F_\theta \otimes_k \mathrm{mf}(R, W) \otimes_R \hat{R} \qquad F_\theta = k\theta_1 \oplus \cdots \oplus k\theta_n$

A_∞ TBD $\quad \mathcal{B} = R/I \otimes_R \mathrm{mf}(R, W)$

A_∞-homotopy equivalence $/k$

$G \circ F \simeq 1$

$$\begin{tikzcd} A \arrow[r] & A \otimes_R \hat{R} \arrow[r, hook] & A_\theta \arrow[loop below, "e"] \arrow[r, shift left, "F"] & \mathcal{B} \arrow[l, shift left, "G"] \arrow[loop below, "E"] \end{tikzcd}$$

homotopy equiv.

$e(\theta_i) = 0$

$E \longleftarrow FeG$

<u>Theorem</u> $(\mathcal{B}, E)$ is an idempotent finite A_∞-model of $A \otimes_R \hat{R}$.

Proof sketch

$$\mathcal{A} = mf(R,W) \qquad R = k[x_1,\dots,x_n]$$
$$\mathcal{A}_\theta = \Lambda F_\theta \otimes_k mf(R,W) \otimes_R \hat{R}$$
$$\mathcal{B} = R/I \otimes_R mf(R,W)$$
$$\begin{tikzcd} \mathcal{A} \arrow[r] & \mathcal{A} \otimes_R \hat{R} \arrow[r, hook] & \mathcal{A}_\theta \arrow[r, shift left, "F"] & \mathcal{B} \arrow[l, shift left, "G"] \end{tikzcd}$$

Choose homotopies λ_i such that

$$[d_{\mathcal{A}}, \lambda_i] = t_i .$$

There is a strict homotopy retraction of complexes over k

$$\begin{tikzcd}
(\mathcal{A}_\theta(X,Y), d_{\mathcal{A}}) = (\Lambda F_\theta \otimes_k \mathrm{Hom}_R(X,Y) \otimes_R \hat{R},\ d_{\mathcal{A}}) \arrow[d, shift left, "e^{-\delta}"] \\
(\Lambda F_\theta \otimes_k \mathrm{Hom}_R(X,Y) \otimes_R \hat{R},\ d_{\mathcal{A}} + \sum_i t_i \theta_i^*) \arrow[u, shift left, "e^{\delta}"] \arrow[d, shift left, "\pi"] \\
(\mathcal{B}(X,Y), \overline{d_{\mathcal{A}}}) = (R/I \otimes_R \mathrm{Hom}_R(X,Y),\ \overline{d_{\mathcal{A}}}) \arrow[u, shift left, "\sigma_\infty"]
\end{tikzcd}$$

$$\delta = \sum_i \lambda_i \theta_i$$

by homological perturbation using connection ∇ → σ_∞

π ← canonical projection

Proof sketch

$$\mathcal{A} = mf(R,W) \quad R = k[x_1,\dots,x_n]$$
$$\mathcal{A}_\theta = \Lambda F_\theta \otimes_k mf(R,W) \otimes_R \hat{R}$$
$$\mathcal{B} = R/I \otimes_R mf(R,W)$$
$$\mathcal{A} \longrightarrow \mathcal{A} \otimes_R \hat{R} \hookrightarrow \mathcal{A}_\theta \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

$$\begin{array}{c} (\mathcal{A}_\theta(X,Y),\, d_{\mathcal{A}}) \\ {\scriptstyle \Phi^{-1}}\ {\scriptstyle e^{\delta} \sigma_\infty} \Big\uparrow \ \text{h.e.} \ \Big\downarrow {\scriptstyle \Phi}\ {\scriptstyle \pi e^{-\delta}} \\ (\mathcal{B}(X,Y),\, \overline{d_{\mathcal{A}}}) \end{array}$$

$$\Phi \circ \Phi^{-1} = 1, \quad \Phi^{-1} \circ \Phi = 1 - [d_{\mathcal{A}}, H]$$

The A_∞-transfer (minimal model) theorem (Kadeshvili, Merkulov, Kontsevich–Soibelman and for our purposes Markl) constructs A_∞-products on $\mathcal{B}$ and A_∞-homotopy equivalences F, G

$$\mathcal{A}_\theta \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

$$F_1 = \Phi, \quad G_1 = \Phi^{-1}, \quad G \circ F \overset{\infty}{\simeq} 1$$

$r_1^{\mathcal{B}}, r_2^{\mathcal{B}}$ induced from $r_1^{\mathcal{A}}, r_2^{\mathcal{A}}$. $\square$

$$\mathcal{A} = mf(R,W) \quad R = k[x_1,\dots,x_n]$$
$$\mathcal{A}_\theta = \Lambda F_\theta \otimes_k mf(R,W) \otimes_R \hat{R}$$
$$\mathcal{B} = R/I \otimes_R mf(R,W)$$
$$\mathcal{A} \longrightarrow \mathcal{A}\otimes_R \hat{R} \hookrightarrow \mathcal{A}_\theta \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

$$(\mathcal{A}_\theta(X,Y), d_{\mathcal{A}}) \underset{\Phi^{-1}}{\overset{\Phi}{\underset{h.e.}{\rightleftarrows}}} (\mathcal{B}(X,Y), \overline{d_{\mathcal{A}}})$$

transfer A_∞-structure

$$\mathcal{A}_\theta(X,Y) = \Lambda F_\theta \otimes_k \mathrm{Hom}_R(X,Y) \otimes_R \hat{R}$$
$$\cong \Lambda F_\theta \otimes_k \mathrm{Hom}_k(\tilde{X},\tilde{Y}) \otimes_k \hat{R}$$

(choose bases for X, Y i.e. $X \cong \tilde{X} \otimes_k R$)

$$\cong \Lambda F_\theta \otimes_k \mathrm{Hom}_k(\tilde{X},\tilde{Y}) \otimes_k R/I \otimes_k k[[t_1,\dots,t_n]]$$
$$\cong \Lambda F_\theta \otimes_k \mathcal{B}(X,Y) \otimes_k k[[t_1,\dots,t_n]] \supset \mathcal{B}(X,Y)$$

$$\nabla = \sum_i \theta_i \frac{\partial}{\partial t_i} \qquad \zeta(\omega\otimes\alpha\otimes f) = \frac{1}{|\omega|+|f|}\,\omega\otimes\alpha\otimes f.$$

$$\mathcal{A} = mf(R,W) \quad R = k[x_1,\dots,x_n] \qquad d_{\mathcal{A}}, m_2$$
$$\mathcal{A}_\theta = \Lambda F_\theta \otimes_k mf(R,W) \otimes_R \hat{R} \qquad d_{\mathcal{A}}, m_2$$
$$\mathcal{B} = R/I \otimes_R mf(R,W) \qquad \overline{d_{\mathcal{A}}}, m_2, m_3, \dots$$
$$\mathcal{A} \longrightarrow \mathcal{A} \otimes_R \hat{R} \hookrightarrow \mathcal{A}_\theta \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{B}$$

$$(\mathcal{A}_\theta(X,Y), d_{\mathcal{A}}) \underset{}{\overset{h.e.}{\rightleftarrows}} (\mathcal{B}(X,Y), \overline{d_{\mathcal{A}}})$$

transfer A_∞-structure

$$\mathcal{A}_\theta(X,Y) \cong \Lambda F_\theta \otimes_k \mathcal{B}(X,Y) \otimes_k k[[t_1,\dots,t_n]] \supset \mathcal{B}(X,Y)$$
$$\nabla = \sum_i \theta_i \frac{\partial}{\partial t_i} \qquad \zeta(\omega \otimes \alpha \otimes f) = \frac{1}{|\omega| + |f|} \omega \otimes \alpha \otimes f.$$

$$At_A := [\nabla, d_A]$$

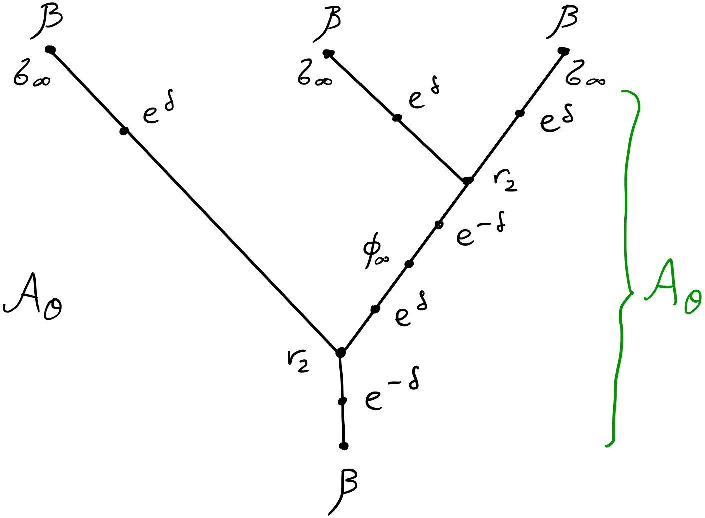
(Atiyah class of $\mathcal{A}$)

$$\sigma_\infty = \sum_{m \geq 0} (-1)^m (\zeta At_A)^m \sigma \; : \; \mathcal{B} \longrightarrow \mathcal{A}_\theta$$

$$\phi_\infty = \sum_{m \geq 0} (-1)^m (\zeta At_A)^m \zeta \nabla \; : \; \mathcal{A}_\theta \longrightarrow \mathcal{A}_\theta$$

$$\delta = \sum_{m \geq 0} \lambda_i \theta_i^* \; : \; \mathcal{A}_\theta \longrightarrow \mathcal{A}_\theta$$

At_A, δ rewritten using tensor T



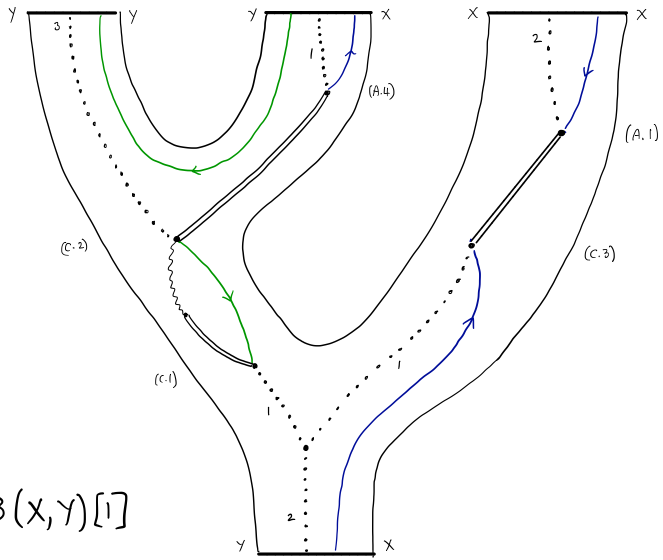
$$r_3 : \mathcal{B}[1]^{\otimes 3} \longrightarrow \mathcal{B}[1]$$

Feynman diagrams

Suppose $X = \bigwedge F_\xi \otimes_k R$, $Y = \bigwedge F_\eta \otimes_k R$ are Koszul-type MFs.

$$\underbrace{\bigwedge(F_\theta \oplus F_\xi^* \oplus F_\eta) \otimes_k R/I \otimes_k k[\![\underline{t}]\!]}_{A_\theta(X,Y),\ \text{interior of trees}} \supset \underbrace{\bigwedge(F_\xi^* \oplus F_\eta) \otimes_k R/I}_{\mathcal{B}(X,Y),\ \text{exterior}}$$

- Apart from ζ all operators involved in computing A_∞-products can be written as polynomials in creation and annihilation operators.
- Feynman diagrams organise reduction of such trees to normal form.



$$r_3(x^2\xi \otimes x\eta\xi^* \otimes x^3\eta^*)$$

<u>Example</u> One contribution for $W = \frac{1}{5}x^5$ to

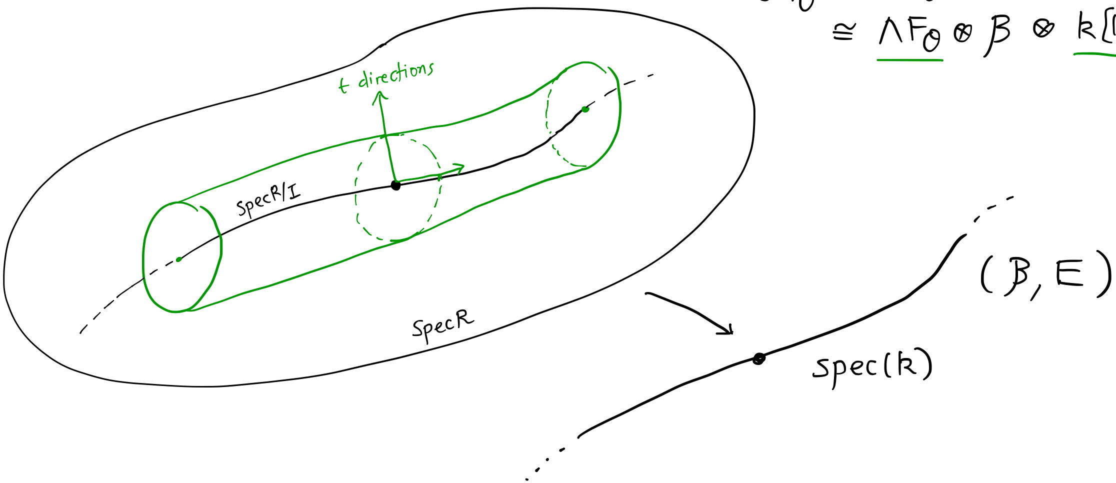
$$r_3 : \mathcal{B}(X,X)[1] \otimes \mathcal{B}(X,Y)[1] \otimes \mathcal{B}(Y,Y)[1] \to \mathcal{B}(X,Y)[1]$$

$$\sigma^* : R/I \otimes_k k[[t_1, \dots, t_n]] \xrightarrow{\cong} \hat{R}$$

$$\nabla : \hat{R} \longrightarrow \hat{R} \otimes_{k[t]} \Omega^1_{k[t]/k}$$

$$\operatorname{Res}_{R/k}\left[\frac{r\, dr_1 \cdots dr_n}{t_1, \dots, t_n}\right] = \operatorname{tr}_{R/I}\left(r[\nabla, r_1] \cdots [\nabla, r_n] \right)$$

$$\mathcal{A}_\theta = \wedge F_\theta \otimes \mathcal{A} \otimes_R \hat{R}$$
$$\cong \underline{\wedge F_\theta} \otimes \mathcal{B} \otimes \underline{k[[t]]}$$



$$r_m^{\mathcal{B}} = r_m^{\mathcal{B}}\left([\nabla, d_A], \lambda_1, \dots, \lambda_n, \zeta \right)$$

Summary

- Want to "compute" $\mathcal{A} = mf(R, W)$, via an idempotent finite model.
- The idempotent finite A_∞-model $(\mathcal{B}, E)$ is a kind of "residue" of $\mathcal{A}$
  1. Choose a Gröbner basis of $I = (t_1, \dots, t_n)$ (e.g. critical locus)
  2. Use this to compute $\hat{R} \cong k[\![\underline{t}]\!] \otimes_k R/I$.
  3. To compute $r_n : \mathcal{B}[1]^{\otimes n} \longrightarrow \mathcal{B}[1]$, E, F, G
     - choose a tree
     - expand $At_{\mathcal{A}}$, e^{δ} terms using 2.
     - reduce to normal form (generates many new terms, indexed by Feynman diagrams, in Koszul cases)
- <u>Hope</u> superpotentials directly from $(\mathcal{B}, E)$.