Spectral sequences for vertex algebras

The aim of this talk is to introduce the relevant material on spectral sequences for Ch. 15 of Frenkel & Ben-Zvi's book on vertex algebras, that is, for the quantum Drinfeld-Sokolov reduction.

The situation is that there is a bicomplex $C = C_k^{\bullet}(\mathfrak{g})_0$ with horizontal differential $\partial_1 = \chi$ (from the Drinfeld-Sokolov character) and vertical differential $\partial_2 = d_{st}$ (semi-infinite cohomology of $\mathfrak{n}_+((t))$), which anticommute $\partial_1\partial_2 + \partial_2\partial_1 = 0$. That is to say,

$$C = \bigoplus_{i,j\in\mathbb{Z}} C^{ij}, \qquad \begin{array}{l} \partial_1 : C^{ij} \longrightarrow C^{i+1,j} \\ \partial_2 : C^{ij} \longrightarrow C^{i,j+1} \end{array} \tag{1.1}$$

The aim is to put a vertex algebra structure on the cohomology of the total complex $(C, \partial_1 + \partial_2)$. This vertex algebra is the W-algebra $W_k(\mathfrak{g})$. To compute this cohomology we use the standard spectral sequence for bicomplexes

$$\underbrace{H^j(H^i(C,\partial_1),\partial_2)}_{\substack{\text{page two of the SS has} \\ \text{these entries + differentials}}} \Longrightarrow \underbrace{H^{i+j}(C,\partial)}_{\substack{\text{page }\infty\text{ of the SS} \\ \text{computes the filter} \\ \text{quotients of this}}} \tag{1.2}$$

<u>Note</u> The $\mathbb{Z}$-grading on the total complex is the obvious one

$$C^n := \bigoplus_{i+j=n} C^{ij} \qquad n = \text{"total degree"} \qquad (i,j) = \text{"bidegree"}$$

In this particular case there is an isomorphism of complexes

$$(C, \partial_1) \cong (V, 0) \otimes (\tilde{C}, \tilde{\partial_1}) \qquad (2.1)$$

(where $V = V(\mathfrak{a}_-)$, $\tilde{C} = \tilde{C}^{\bullet}_k(\mathfrak{g})_0$ in the book) and a filtration of $\tilde{C}$ by subcomplexes $\{F^p\tilde{C}\}_{p\leq 0}$, with the property that

decreasing filtration

$$① \quad H^\bullet\left(F^p\tilde{C} / F^{p+1}\tilde{C} \right) \cong \begin{cases} \mathbb{C} \text{ in bidegree } (0,0) & p = 0 \\ 0 & p \neq 0 \end{cases}$$

Note: $F^p\tilde{C}$ is spanned by monomials of length $\leq -p$ applied to $|0\rangle$. We use the grading by total degree on $\tilde{C}$, so $H^0(F^0\tilde{C})$ means elements of total deg 0 in the kernel, mod boundaries.

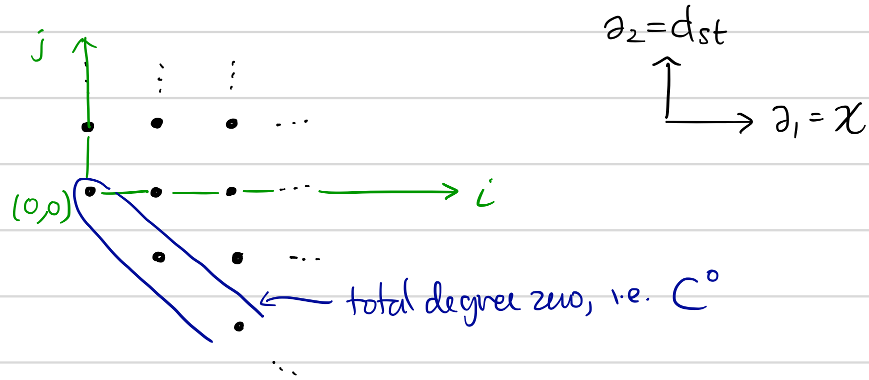
The next step is to deduce:

② $H^\bullet(\tilde{C}, \tilde{\partial_1}) \cong \mathbb{C}$ using the spectral sequence associated to a filtered complex and ①.

Note Again, $\tilde{C}^0 \subseteq C^0 = \bigoplus_{i+j=0} C^{ij}$ and the generator of the above cohomology lives in C^{00}. It's important that $C^{ij} = 0$ for $i<0$ and $i+j<0$, i.e. the total degree is always ≥ 0

A diagram of the nonzero bidegrees of C is therefore

$\boxed{C}$



(3.1)

We are told that $V \subseteq \mathrm{Ker}(\partial_1)$ and $V \subseteq C^0$, so while elements of V may have bidegree $\neq (0,0)$ they all have <u>total degree</u> zero.

Next, since (2.1) is an isomorphism of complexes identifying <u>bidegrees</u> on the left and right (this is clear from the construction),

3 $\quad H^\bullet(C, \partial_1) = V$ using 2 and (2.1), meaning that

$$H^\bullet\big(C^0 \xrightarrow{\partial_1} C^1 \xrightarrow{\partial_1} C^2 \xrightarrow{\partial_1} \cdots \big) = V \quad 0 \quad 0 \quad \cdots \qquad (3.2)$$

means total deg

equals, not $\cong$, since the iso is $V \hookrightarrow \mathrm{Ker}(\partial_1) = H^0$

total degree zero.

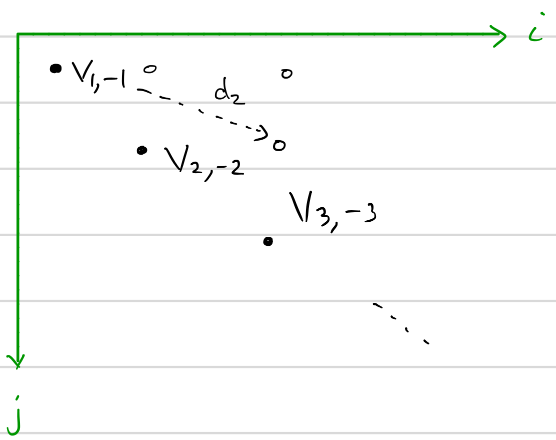
4 The spectral sequence has the left hand side as its E_2 page. The confusing thing is that H^i there refers to the (horizontal) i-grading only, (resp H^j), compared to the $H^\bullet$ above in 3 which is total degree.

In any case, the calculation of (3.2) shows that

$$\begin{aligned} H^i(C, \partial_1) &= \bigoplus_j H^i\big(\cdots \to C^{i-1,j} \xrightarrow{\partial_1} C^{ij} \xrightarrow{\partial_1} C^{i,j+1} \to \cdots \big) \\ &= \mathrm{Ker}\big(C^{i,-i} \xrightarrow{\partial_1} C^{i+1,-i} \big) \\ &= V^{i,-i} \quad \text{(i.e. the bidegree } (i,-i) \text{ piece of } V \subseteq C\text{).} \end{aligned}$$

But then clearly the induced map by ∂_2 is zero, so the E_2 page is identically zero, except along the diagonal $i+j=0$ where it is $V_{i,-i}$.

$\boxed{E_2}$



(4.1)

Differentials on p.2 of a spectral sequence go across 2 and down 1, as shown, so the spectral sequence has $d_2 = 0$, i.e. it degenerates on p.2. We conclude that $H^n(C,\partial) = 0$ unless $n=0$, and

$$Wh(\mathfrak{g}) := H^0(C,\partial) \underset{(*)}{\cong} V_{1,-1} \oplus V_{2,-2} \oplus \cdots = V. \qquad (4.2)$$

The isomorphism $(*)$ however is nontrivial. What the spectral sequence tells us is that the canonical filtration of $H^0(C,\partial)$ given by the images F^p of the canonical maps

$$\begin{tikzcd} H^0\left(\bigoplus_{r\geq p} C^{-r,r}, \partial\right) \arrow[rr, "can"] \arrow[dr, two heads] & & H^0(C,\partial) \\ & F^p \arrow[ur, hook] & \end{tikzcd} \qquad (4.3)$$

Has the property that (a) it is a decreasing filtration $F^p \supseteq F^{p+1}$ and

$$F^p/F^{p+1} \cong E_\infty^{p,-p} \cong E_2^{p,-p} = V_{p,-p} \qquad (4.4)$$

Introduction to spectral sequences

Now we give a sketch of how one constructs the spectral sequences used in the above (recall: we used the SS of a filtered complex, and the SS of a bicomplex, which as it turns out is a special case of the first SS). Our reference is

[therisingsea.org / notes / Spectral Sequences.pdf]

Let $\mathcal{A}$ be an abelian category and $\{F^pC\}_{p\in\mathbb{Z}}$ a decreasing (i.e. $F^pC \supseteq F^{p+1}C$) chain of subcomplexes of a cochain complex C in $\mathcal{A}$. The inclusions $\iota_p : F^pC \longrightarrow C$ induce morphisms

$$H^*(\iota_p) : H^*(F^pC) \longrightarrow H^*(C)$$

and we define

$$E^n := H^nC, \qquad F^pE^n := \operatorname{Im}(H^n(\iota_p)) \subseteq E^n$$

Then clearly this gives a filtration of E^n by subobjects $\{F^pE^n\}_{p\in\mathbb{Z}}$ and the question answered by the spectral sequence associated to the filtration $\{F^pC\}_p$ is the following:

$\boxed{Q}$ Can we "compute" the filter quotients $F^pE^n/F^{p+1}E^n$ from the filter quotients $F^pC/F^{p+1}C$?

With mild finiteness conditions on C, the answer is <u>Yes</u>, but the nature of the "computation" is quite elaborate.

Outline

- Define A_r^{pq}, $\ddot{A}_r^{pq}$
- Define Z_r^{pq}, B_r^{pq}, E_r^{pq}
- Define d_r^{pq}
- Define $Z_{r+1}(E_r^{pq})$, $B_{r+1}(E_r^{pq})$
- Define $Z_k(E_r^{pq})$, $B_k(E_r^{pq})$ (inc $k=\infty$)
- Define E_∞ and show $E_\infty^{pq} \cong gr_p(E^{p+q})$
- Discuss E_0
- Define biregularity, first quadrant filtration
- Degeneration on page r