tsupera

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5/5/16

The super-A-polynomial and knot differentials

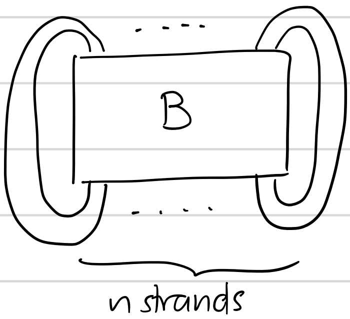
The aim of this talk is to explain what the super-A-polynomial and its quantised version are supposed to be, various generalisations of the volume conjecture and how homological knot invariants complicate the relationships among these conjectures.

1. HOMFLY homology
2. Coloured HOMFLY homology
3. Recurrence relations and the (quantised, super) A-polynomial
4. Results on knot differentials

<u>Volume conjecture</u> The super-A-polynomial encodes "colour dependence" of the coloured HOMFLY homology via recurrence rel^{N}s.

1. HOMFLY homology (Khovanov–Rozansky) [KR, K]

Knot K = closure of braid B



$P = \mathbb{Q}[x_1, \dots, x_n]$ $\quad |x_i| = 2$

$B \longmapsto$ complex of graded P-P-bimodules $F^*(B)$

(i.e. $P^e = P \otimes_{\mathbb{Q}} P$-modules)

$$HHH(K) := H^* \operatorname{Tor}_*^{P^e}(P, F^*(B)) \cong \overline{HHH}(K) \otimes_{\mathbb{Q}} \mathbb{Q}[a]$$

triply graded

also Rouquier

<u>Theorem</u> (Khovanov-Rozansky) $HHH(K)$ is a knot invariant and the superpolynomial

$$P(K;a,q,t) = \sum_{i,j,k} t^i q^j a^k \dim_{\mathbb{Q}} \overline{HHH}_{i,j,k}(K)$$

has the property that

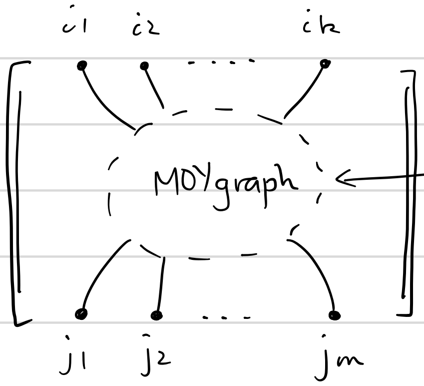
$$P(K;a,q,t)\big|_{t=-1} = \text{HOMFLY}(K).$$

(i = homological, j = polynomial, k = Tor/Hochschild)

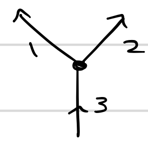
<u>Construction</u> (following [MSV]) $(\otimes = \otimes_{\mathbb{Q}})$

$$\Big[\,\overset{i_1}{\bullet}\;\overset{i_2}{\bullet}\;\cdots\;\overset{i_k}{\bullet}\,\Big] = \underbrace{\mathbb{Q}[\mathbb{X}_1]^{S_{i_1}} \otimes \mathbb{Q}[\mathbb{X}_2]^{S_{i_2}} \otimes \cdots \otimes \mathbb{Q}[\mathbb{X}_k]^{S_{i_k}}}_{\text{call this } R_{i_1,\dots,i_k}}$$

$|\mathbb{X}_p| = i_p$ variables, deg 2



oriented trivalent graph with conservative edge weights $\in \mathbb{N}$ e.g.

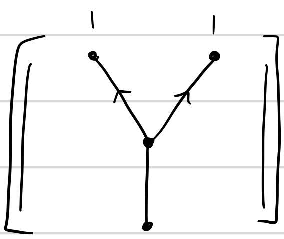


$\|$

$R_{i_1,i_2,\dots,i_k} - R_{j_1,j_2\cdots j_m}$-bimodule (graded)

Example

$\mathbb{Q}[x_1, x_2]$



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$\mathbb{Q}[x_1,x_2]^{S_2} = \mathbb{Q}[x_1+x_2, x_1x_2]$

$:=$ $\mathbb{Q}[x_1, x_2]$ as a $\mathbb{Q}[x_1,x_2]$-$\mathbb{Q}[x_1,x_2]^{S_2}$-bimodule

(tensoring gives <u>inclusion</u> $\mathrm{Mod}\,\mathbb{Q}[x_1,x_2]^{S_2} \longrightarrow \mathrm{Mod}\,\mathbb{Q}[x_1,x_2]$)

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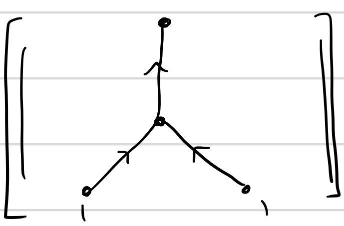
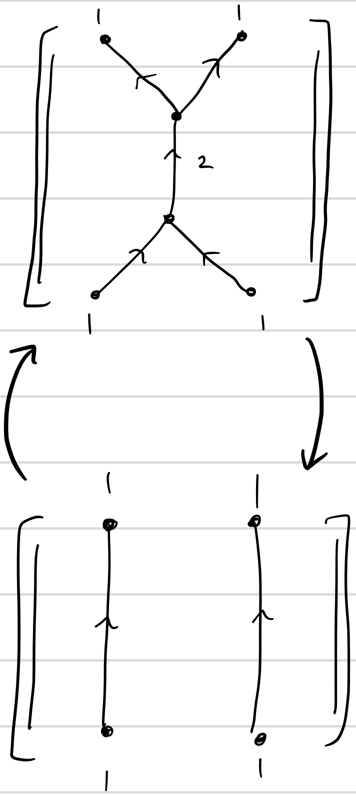


$:=$ $\mathbb{Q}[x_1, x_2]$ as a $\mathbb{Q}[x_1,x_2]^{S_2}$-$\mathbb{Q}[x_1,x_2]$-bimodule

(tensoring gives <u>restriction</u> $\mathrm{Mod}\,\mathbb{Q}[x_1,x_2] \longrightarrow \mathrm{Mod}\,\mathbb{Q}[x_1,x_2]^{S_2}$)

Any MOY graph with external weights $\equiv 1$ has its bimodule constructed from these by $\otimes$:



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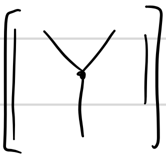
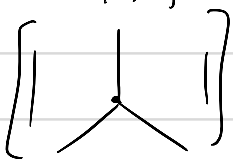
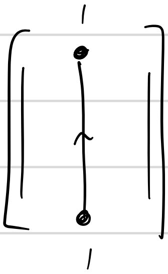


$\otimes_{\mathbb{Q}[x_1,x_2]^{S_2}}$

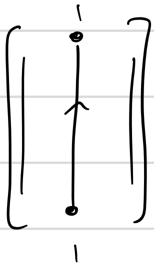


$= \mathbb{Q}[x_1,x_2] \otimes_{\mathbb{Q}[x_1,x_2]^{S_2}} \mathbb{Q}[x_1,x_2]$

$=$



$\otimes_{\mathbb{Q}}$



$= \mathbb{Q}[x_1] \otimes_{\mathbb{C}} \mathbb{Q}[x_2]$

$\cong \mathbb{Q}[x_1, x_2]$.

<u>Fact</u> Bimodules $[\![Y]\!]$, $[\![\lambda]\!]$ are an adjoint pair, in both directions

(i.e. $[\![Y]\!] \dashv [\![\lambda]\!]$ <u>and</u> $[\![\lambda]\!] \dashv [\![Y]\!]$)

$$M\{k\}_i = M_{i+k}$$

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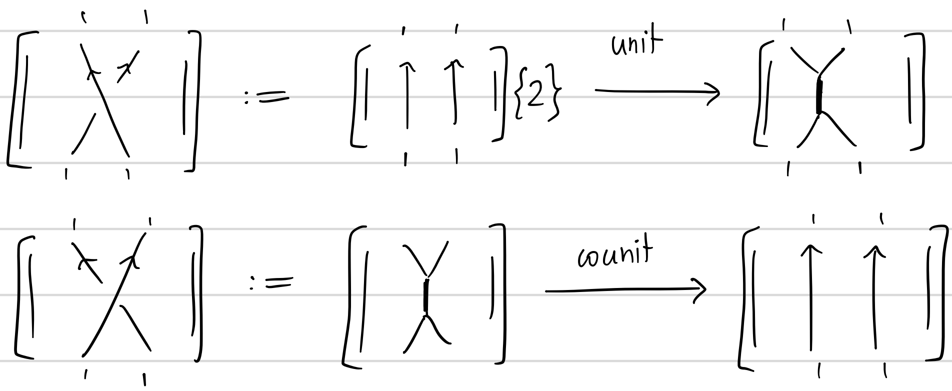
(4)



$b \mapsto \frac{1}{2}(x_1 - x_2) \otimes b + 1 \otimes \frac{1}{2}(x_1 - x_2) b$

$a \otimes b \mapsto ab$

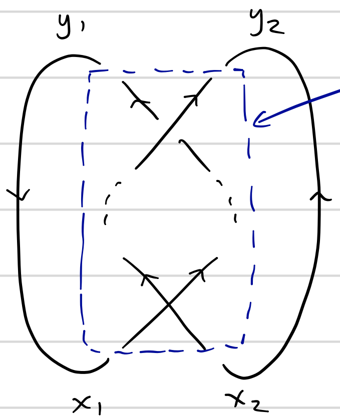
homological degree zero

Tensoring these complexes horizontally and vertically defines (using external edge weights $\equiv 1$)

$$B \longmapsto F^*(B) \quad \text{complex of graded } P\text{-}P\text{-bimodules.}$$

<u>Example</u> $(2,p)$ torus knot $T_{2,p}$ ($p=3$ trefoil)

$T_{2,p} =$



braid B has p 's

<u>Lemma</u> $F^*(B) = S\{p\} \longrightarrow \cdots \xrightarrow{x_1 - y_1} S\{2\} \xrightarrow{x_1 - y_2} S\{1\} \xrightarrow{x_1 - y_1} S \xrightarrow{\text{can}} \underbrace{P}$

i.e. multiplication

homological degree zero

$$S := \mathbb{Q}[x_1, x_2] \underset{\mathbb{Q}[x_1,x_2]^{S_2}}{\otimes} \mathbb{Q}[x_1, x_2] \cong \frac{\mathbb{Q}[x_1, x_2, y_1, y_2]}{(e_1(y) - e_1(x),\ e_2(y) - e_2(x))}$$

$$P = \mathbb{Q}[x_1, x_2]$$

This is a complex since

$$\begin{aligned}(x_1-y_1)(x_1-y_2) &= x_1^2 - x_1y_2 - y_1x_1 + y_1y_2 \\ \text{over } S \quad &= x_1^2 - x_1y_2 - y_1x_1 + x_1x_2 \\ &= x_1^2 - x_1(y_1+y_2) + x_1x_2 \\ &= x_1^2 - x_1\cdot(x_1+x_2) + x_1x_2 = 0.\end{aligned}$$

Then by def^N, $HHH(T_{2,p})$ is the homology of

$$\left(HH_i(-) := \mathrm{Tor}_i^{P^e}(P, -) \text{ is } \underline{\text{Hochschild homology}} \right) \qquad (*)$$

$$HH_*(S\{p\}) \to \cdots \xrightarrow{x_1-y_2} HH_*(S\{1\}) \xrightarrow{x_1-y_1} HH_*(S) \longrightarrow \underset{\boxed{0}}{HH_*(P)}$$

To compute this:

$$HH_*(S) = H_*\left(P \overset{\mathbb{L}}{\underset{P^e}{\otimes}} S\right) = H_*\left(P \underset{P^e}{\otimes} \left[\begin{array}{c} P \\ \uparrow \scriptstyle (e_1(y)-e_1(x),\ e_2(y)-e_2(x)) \\ P^{\oplus 2} \\ \uparrow \scriptstyle \begin{pmatrix} -[e_2(y)-e_2(x)] \\ e_1(y)-e_1(x) \end{pmatrix} \\ P \end{array} \right] \right)$$

$$= H_*\left(\begin{array}{c} P \\ \uparrow 0 \\ P^{\oplus 2} \\ \uparrow 0 \\ P \end{array} \right)$$

$\therefore$ (*) is the complex

$$\left[\begin{array}{ccccccccc} \xrightarrow{0} & P & \xrightarrow{x_1-x_2} & P & \xrightarrow{0} & P & \xrightarrow{1} & P \\ \xrightarrow{0} & P^{\oplus 2} & \xrightarrow{x_1-x_2} & P^{\oplus 2} & \xrightarrow{0} & P^{\oplus 2} & \xrightarrow{\begin{pmatrix} 1 & x_2 \\ 1 & x_1 \end{pmatrix}} & P^{\oplus 2} \\ \xrightarrow{0} & P & \xrightarrow{x_1-x_2} & P & \xrightarrow{0} & P & \xrightarrow[x_1-x_2]{} & P \end{array} \right]$$

Which has cohomology

polynomial (q) j ; homological (t) i ; Hochschild (a) k

$$\begin{array}{cccccc} & & & & & \\ \left[\begin{matrix} \cdots & 0 & \mathbb{Q}[x] & 0 & 0 \\ \cdots & 0 & \boxed{\mathbb{Q}[x]^{\oplus 2}} & 0 & \mathbb{Q}[x] \\ \cdots & 0 & \mathbb{Q}[x] & 0 & \mathbb{Q}[x] \end{matrix}\right] \\ \leftarrow \cdots & & -2\ \{1\} & -1 & 0 \end{array}$$

up to $-(p+1)$

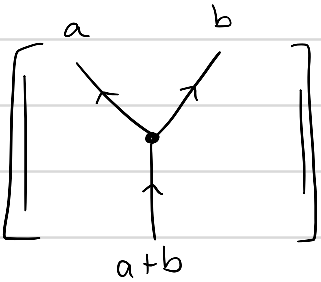
$$HHH(T_{2,p}) \cong \overline{HHH}(T_{2,p}) \otimes_{\mathbb{Q}} \mathbb{Q}[x].$$

e.g. $\overline{HHH}_{-2,1,1} \cong \mathbb{Q}^{\oplus 2}$

$\mathbb{Q}^{\oplus 2}\{1\}$

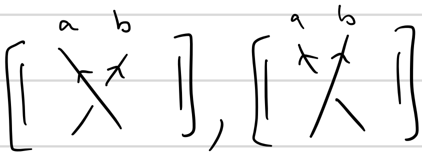
② Coloured HOMFLY homology (Mackaay-Stosic-Vaz, Webster-Williamson)

Similar construction, but with arbitrary labels (above was all 1's)



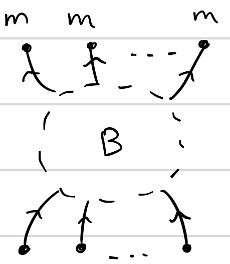
$$= \mathbb{Q}[y_1, \ldots, y_a, z_1, \ldots, z_b]^{S_a \times S_b}$$

$\left(\text{as } \mathbb{Q}[y_1, \ldots, y_a, z_1, \ldots, z_b]^{S_a \times S_b} - \mathbb{Q}[x_1, \ldots, x_{a+b}]^{S_{a+b}}\right.$ graded bimodule $\Big)$



= some complexes of these bimodules (more complicated!)

$$HHH^{(m)}(K) := HH_* \left[\!\left[\right.\right.$$



$$\underbrace{\left.\left. \right]\!\right]}_{\substack{\text{complex of graded} \\ \mathbb{Q}[\mathbb{X}_1]^{S_m} \otimes \cdots \otimes \mathbb{Q}[\mathbb{X}_n]^{S_m}\text{- bimodules}}} \cong \overline{HHH}^{(m)}(K) \otimes_{\mathbb{Q}} \mathbb{Q}[a].$$

$$P_m(K; a, q, t) = \sum_{i,j,k} t^i q^j a^k \dim_{\mathbb{Q}} \overline{HHH}^{(m)}(K)$$

<u>Theorem</u> (MSV, WW) $P_m(K; a, q, t)\big|_{t=-1}$ = m-coloured HOMFLY.

$$\begin{tikzcd}
\underset{\text{coloured HOMFLY poly}}{P_m(K; a, q)} \arrow[r, shift left, "\text{categorify}"] \arrow[d, "a=q^N"'] & \underset{\text{coloured HOMFLY homology}}{P_m(K; a, q, t)} \arrow[l, shift left, "t=-1"] \arrow[d, dashed, "\text{differential } d_N"] \\
\underset{\text{coloured Jones}}{P_m^{sl(N)}(K; q)} \arrow[r, shift left, "\text{categorify}"] & \underset{\text{coloured sl(N) Khovanov–Rozansky homology}}{P_m^{sl(N)}(K; q, t)} \arrow[l, shift left, "t=-1"]
\end{tikzcd}$$

3 The super-A-polynomial

$$\hat{A}^{super}(\hat{x},\hat{y};a,q,t) \in \frac{\mathbb{Q}[\hat{x},\hat{y},a^{\pm 1},q^{\pm 1},t^{\pm 1}]}{(\hat{x}\hat{y}-q\hat{y}\hat{x})}$$

quantum super-A-polynomial

$$\boxed{\begin{aligned}\hat{x}P_m &= q^m P_m \\ \hat{y}P_m &= P_{m+1}\end{aligned}}$$

$$\circlearrowright$$

$$\mathbb{Q}[\{P_m\}_{m\geq 1}, q^{\pm 1}]$$

$$\Big\downarrow \varphi_K \qquad \varphi_K(P_m) = P_m(K;a,q,t)$$

$$\mathbb{Q}[a^{\pm 1},q^{\pm 1},t^{\pm 1}]$$

$\hat{A}^{super}$ is supposed to be defined by some BPS counting which I do not understand, but this is easy enough to appreciate:

<u>Homological quantum volume conjecture</u>

For any knot K, and $m \geq 1$, $\varphi_K(\hat{A}^{super}\cdot P_m) = 0$

i.e. if $\hat{A}^{super} = \sum_i a_i(\hat{x},a,q,t)\hat{y}^i$ then

$$\cdots + a_1 P_{m+1}(K) + a_0 P_m(K) = 0.$$

recurrence relation

$$A^{super} := \hat{A}^{super}\big|_{q=1}$$

$$A^{super}(x,y;a,t)\big|_{t=-1,a=1} = A(x,y)$$

classical A-polynomial

probably comes down to understanding "colour differentials" (?)

(e.g. [FGS] p.30)
[FGS2] §4

4 Knot differentials

Conjecture (Dunfield, Gukov, Rasmussen) "DGR differentials"

For $N \geqslant 2$ $\exists\, d_N : HHH(K) \longrightarrow HHH(K)$, $d_N^2 = 0$

$$H^*(HHH(K), d_N) \cong \text{Khovanov-Rozansky } sl(N)\text{-homology of } K. \quad KR_N(K)$$

triply graded

bigraded

$\begin{array}{l} t, q, a \\ i, j, k \end{array}$

$\ulcorner\, |d_N| = (0, N+1, -1) \,\lrcorner$

Example $HHH(T_{2,p})$ was computed by

$$\begin{bmatrix}\begin{tikzcd}
\cdots & 0 & \mathbb{Q}[x] \arrow[d, dashed, "{\begin{pmatrix}x^N\\ x^{N-1}\end{pmatrix}=d_N}"'] & 0 & 0 \\
\cdots & 0 & \mathbb{Q}[x]^{\oplus 2} \arrow[d, dashed, "{(-x^{N-1}\ \ x^N)=d_N}"'] & 0 & \mathbb{Q}[x] \arrow[d, dashed, "{x^N=d_N}"'] \\
\cdots & 0 & \mathbb{Q}[x] & 0 & \mathbb{Q}[x]
\end{tikzcd}\end{bmatrix}$$

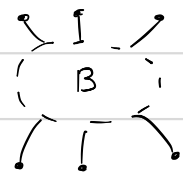
s.t. $H^*(HHH(T_{2,p}), d_N) \overset{\checkmark}{\cong} KR_N(T_{2,p})$

$$\| $$

$$\begin{bmatrix}
\cdots & 0 & 0 & 0 & 0 \\
\cdots & 0 & \mathbb{Q}[x]/x^{N-1} & 0 & 0 \\
\cdots & 0 & \mathbb{Q}[x]/x^{N-1} & 0 & \mathbb{Q}[x]/x^N
\end{bmatrix}$$

Origin of differentials (joint w/ Rouquier)

n strands

$$HHH_*(K) = HH_* \Big[\!\Big[$$  $$\Big]\!\Big]$$

complex of P-P-bimodules $F^*(B)$

$P = \mathbb{Q}[x_1, \dots, x_n]$

$P^e = P \otimes_{\mathbb{Q}} P$

With $S = P^e / (e_i \otimes 1 - 1 \otimes e_i)$

$F^*(B)$ is a complex of graded S-modules

$$\therefore \quad \begin{tikzcd} D(S\text{-Mod}) \arrow[r, "(-)_{P^e}"] \arrow[rr, bend right, "\Phi := P \overset{\mathbb{L}}{\otimes}_{P^e} (-)_{P^e}"'] & D(P^e\text{-Mod}) \arrow[r, "P \overset{\mathbb{L}}{\otimes}_{P^e} -"] & D(P\text{-Mod}) \end{tikzcd}$$

$$\Phi(F^*(B)) = P \overset{\mathbb{L}}{\otimes}_{P^e} F^*(B)$$

$$\therefore H^* \Phi(F^*(B)) = HHH(K)$$

$\circlearrowleft \delta_N$ $\qquad$ $\circlearrowleft d_N$

natural transformation

$\delta_N : \Phi \longrightarrow \Phi\{?\}[?]$

<u>Theorem</u> (M-Rouquier) There is an isomorphism of graded rings

$$\Lambda V^* \underset{\mathbb{Q}}{\otimes} \Lambda U \underset{\mathbb{Q}}{\otimes} P \xrightarrow[Q]{\cong} \mathrm{End}^*(\Phi).$$

$$V = \mathbb{Q}\theta_1 \oplus \cdots \oplus \mathbb{Q}\theta_n, \quad U = \mathbb{Q}\xi_1 \oplus \cdots \oplus \mathbb{Q}\xi_n$$

Given $W \in P^{S_n}$ we define

$$\begin{array}{ccc} \Lambda U \otimes P^{S_n} & \longrightarrow & \mathrm{End}^*(\Phi) \\ \cup & & \cup \\ \rho_W = \sum_i \partial_{e_i}(W)\xi_i & \longmapsto & \text{def } \delta_W := Q(\rho_W) \end{array}$$

<u>Conjecture</u> $d_N := \delta_W$ for $W = \sum_i x_i^{N+1}$ is the DGR differential

<u>Theorem</u> Works for $(2,n)$-torus knots, i.e. for $W = \sum_i x_i^{N+1}$

$$H^*(HHH(K), \delta_W) \cong KR_N(K).$$

$$K = T_{2,n}$$

References

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