mbcrit
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Monoidal (bi) categories of critical points

Part I : Review of non-linear dynamics

Why do we care about critical points of functions? Recall that the differential equation $m\ddot{x} = F(x, \dot{x})$ governing the motion $x = x(t)$ of a particle can be written as a system of DEs

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \tfrac{1}{m} F(x, y) \end{aligned}$$

which centers the analysis of dynamics on the vector field $(y, \frac{1}{m}F(x,y))$ in the phase plane. If the force F depends explicitly on time, then this vector field also depends on time. A general non-linear dynamical system in n-dimensions is given by a system of DEs

$$\left.\begin{aligned} \dot{x}_1 &= f_1(x_1, \dots, x_n) \\ \dot{x}_2 &= f_2(x_1, \dots, x_n) \\ &\;\;\vdots \\ \dot{x}_n &= f_n(x_1, \dots, x_n) \end{aligned}\right\} \qquad \begin{aligned} \dot{\underline{x}} &= f(\underline{x}) \\ f &: \mathbb{R}^n \longrightarrow \mathbb{R}^n \end{aligned}$$

and central to its analysis is the corresponding vector field on $\mathbb{R}^n$. Once again, we are interested in one-parameter families of such vector fields, or just infinitesimal perturbations, because those are physically natural. It is typically impossible to solve such a system (see e.g. Strogatz "Nonlinear dynamics and chaos") but nonetheless a good qualitative understanding of the phase portrait is still possible, and this understanding is organised by

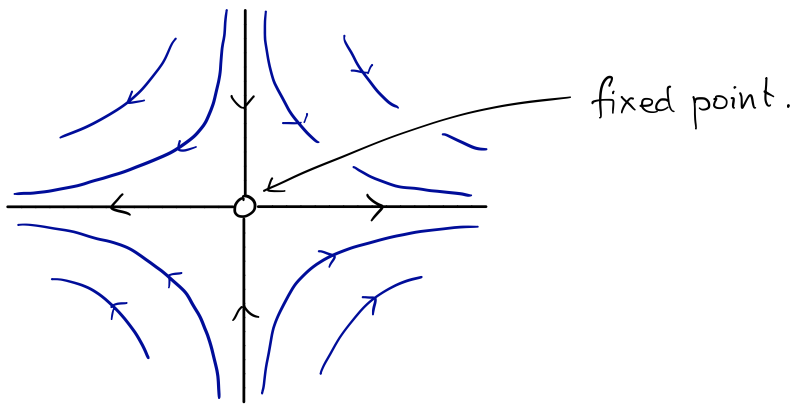
- fixed points i.e. $\underline{x} \in \mathbb{R}^n$ such that $f(\underline{x}) = \underline{0}$
- closed orbits and limit cycles.

we concentrate on fixed points

<u>Example</u> Consider the system

$$\dot{x}_1 = x_1 \qquad\qquad \underline{\dot{x}} = \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}^{A} \underline{x}$$
$$\dot{x}_2 = -x_2$$

Of course we know solution trajectories to this look like $\underline{x}(t) = (Ae^{t}, Be^{-t})$ for any $A, B \in \mathbb{R}$ and the phase portrait is



The information in the phase portrait is completely captured by the fixed point, and the eigenvectors and eigenvalues of A.

For a general n-dimensional system, the behaviour near any isolated fixed point can be analysed in the same way using the Jacobian of F at the fixed point, and studying its eigenvalues and eigenvectors. An important class of dynamical systems are those which are conservative, in the sense that there is a <u>scalar potential</u> $f : U \longrightarrow \mathbb{R}$ with $U \subseteq \mathbb{R}^n$,

$$F = \nabla f.$$

At least if U is simply connected, such a potential exists iff. F is irrotational, i.e. $\partial F_i / \partial x_j = \partial F_j / \partial x_i$ for all $i \neq j$ (since $H^1_{dR}(U) = 0$).

<u>Example</u> $f_{1,1}(x_1,x_2) = x_1^2 - x_2^2$ gives the system of the previous example.
In general we define for $p+q=n$

$$f_{p,q}(x_1,\dots,x_n) = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_n^2$$

which gives a dynamical system in which p directions "repel" solutions from the origin and q directions "attract".

Notice that

$$\{\text{ fixed points of system }\} = \{\text{ critical points of } f \}$$

$$\nabla f(\underline{x}) = 0$$

and to understand (to a first degree of approximation) the dynamics near an isolated critical point we need to analyse the <u>Hessian</u> of f, i.e.

$$H_f := \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{1 \le i,j \le n},$$

its eigenvectors and eigenvalues. Actually the right way to think of this data is as a symmetric bilinear form on the tangent space $T_{\underline{x}^*}U$ at a critical point $\underline{x}^* \in U$, i.e.

$$(T_{\underline{x}^*}U, \langle , \rangle) \quad \text{where} \quad \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle = \left. \frac{\partial^2 f}{\partial x_i \partial x_j} \right|_{\underline{x} = \underline{x}^*}$$

Such a pair is called a <u>quadratic space</u>.

Let $\underline{x}^* \in U$ be a critical point. Then with $\underline{u} = \underline{x} - \underline{x}^*$ we get

$$\underbrace{\dot{\underline{u}} = H_f\big|_{\underline{x}^*}\,\underline{u}}_{\text{linear system}} + \text{quadratic terms in } \underline{u} \text{ involving higher derivatives of } f \tag{4.1}$$

To what extent is this linear system actually a good match for the local dynamics of the original non-linear system?

<u>Morse Lemma</u> If $H_f\big|_{\underline{x}^*}$ is invertible (i.e. the corresponding bilinear form is nondegenerate) then there is a coordinate neighborhood around $\underline{x}^*$ where $f = f_{p,q}$ for some $p+q=n$, so

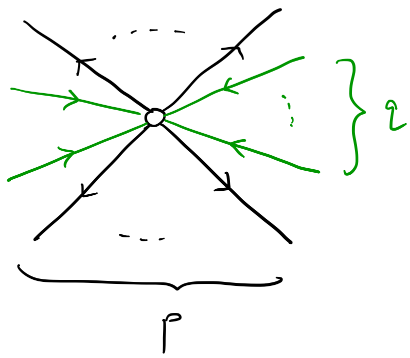
$$H_f\big|_{\underline{x}^*} = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{pmatrix}$$
$$\qquad\qquad\qquad\qquad\underbrace{\qquad\quad}_{p}\ \underbrace{\qquad\quad}_{q}$$

In this case there are no higher derivatives, and the local dynamics are completely captured by the quadratic space $(T_{\underline{x}^*}U,\ H_f)$, which in turn are classified by their <u>signature</u> (p,q) (Sylvester's law of inertia).

<u>Def^n</u> A critical point $\underline{x}^*$ is <u>nondegenerate</u> if $H_f|_{\underline{x}^*}$ is invertible.

Moreover, every degenerate critical point "bifurcates" into some nondegenerate points after an arbitrarily small perturbation (e.g. x^3 has a degenerate critical point at $x^* = 0$, while $x^3 - \lambda x$ for $\lambda > 0$ has nondegenerate critical points at $\pm\sqrt{\lambda}/\sqrt{3}$) so generically every critical point is nondegenerate. (for a good telling of this story see Arnold's ICM address "Critical points of smooth functions" from 1974).

<u>Upshot</u>: if our dynamical system has a scalar potential, and we are willing to infinitesimally perturb the system, the dynamics near critical points are easily understood, and look like:



<u>However</u> If we consider a time varying potential, and vector field, there may be no perturbation of the <u>family</u> of potentials which "bifurcates" a degenerate singularity at time $t=0$ (e.g. $x^3 - tx$), and so degenerate singularities are also of physical interest. In such cases the local dynamics involves higher terms in (4.1).

<u>Part II</u>: <u>A category of critical points</u>

"critical points should communicate"

Restricting for the moment to nondegenerate critical points, can these be viewed as objects of a <u>category</u>? Equivalently, is there a natural notion of <u>morphism</u> between critical points? Well, clearly it makes sense to talk about <u>symmetries</u> of critical points (e.g. $x_1 \longleftrightarrow -x_2$ in $x_1^2 + x_2^2 - x_3^2$), and thus about local symmetries of trajectories near the critical point, but perhaps it is not clear there are any other interesting morphisms, a priori.

Nonetheless, let us proceed with an analysis of symmetries and see what turns up.

The correct notion of symmetry of the critical point $\underline{x}^*$ of $f: U \to \mathbb{R}$ is an isometry of the quadratic pair $(V, \langle , \rangle) := (T_{\underline{x}^*}U, H_f)$ where

$$\langle u, v \rangle = u^T H_f v.$$

That is, a linear map $T: V \to V$ (necessarily iso) with $\langle Tu, Tv \rangle = \langle u, v \rangle$. This suggests that we consider

<u>Def</u>

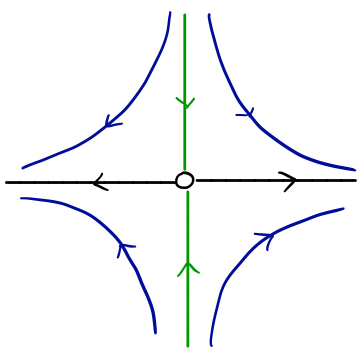
Def^n The category $\mathcal{Q}$ of quadratic spaces has

- <u>objects</u> are f.d. vector spaces equipped with a nondegenerate symmetric bilinear form.
- <u>morphisms</u> $\mathrm{Hom}_{\mathcal{Q}}(V, W) = \{ T: V \to W \text{ linear} \mid \langle Tu, Tv \rangle = \langle u, v \rangle \ \forall u, v \}$.

<u>Example</u> $X_{1,0} = \left(\mathbb{R}\frac{\partial}{\partial x_1}, (1) \right) \longrightarrow \left(\mathbb{R}\frac{\partial}{\partial x_1} \oplus \mathbb{R}\frac{\partial}{\partial x_2}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right) =: X_{1,1}$

$\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} \rangle = 1$ $\qquad$ $\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} \rangle = 1$, $\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2} \rangle = -1$

This morphism corresponds to viewing the "black" dynamics as a subset of the overall phase portrait given below. If we write $X_{p,q}$



for the quadratic space with signature p, q then there is also a morphism $X_{0,1} \to X_{1,1}$. However it is not the case that $X_{1,1} \cong X_{0,1} \oplus X_{1,0}$, since there is no morphism $X_{1,1} \to X_{1,0}$!

⌜since if $T(\frac{\partial}{\partial x_2}) = \alpha \frac{\partial}{\partial x_1}$ we'd need

$$-1 = \langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2} \rangle = \langle \alpha \frac{\partial}{\partial x_1}, \alpha \frac{\partial}{\partial x_1} \rangle = \alpha^2 \text{⌟}$$

<u>Lemma</u> Q is a monoidal category under <u>direct sum</u> of v. spaces, with unit $0 = X_{0,0}$.

(to tensor two quadratic spaces we <u>add</u> their potentials, in disjoint variable sets)

<u>Example</u> $X_{1,0} \otimes X_{0,1} \cong X_{1,1}$, and more generally $X_{p,q} \otimes X_{p',q'} \cong X_{p+p',q+q'}$.

Clearly Q is generated as a monoidal category (i.e. under $\otimes$) by the quadratic spaces $X_{1,0}$ and $X_{0,1}$. Actually Q is a monoidal category with <u>duals</u> and $X_{1,0}^{\vee} = X_{0,1}$. So Q is generated, as a monoidal category with duals, by one object. So that's amusing, but this isn't so far a compelling case that there is any deep content to considering morphisms of quadratic spaces beyond isometries.

<u>Nonetheless</u> let us persist. Associated to each quadratic space V is an algebra $C(V)$, the <u>Clifford algebra</u>, which is universal among $\mathbb{R}$-algebras C (associative and unital) equipped with a linear map $\iota : V \longrightarrow C$ satisfying

$$\iota(v)\iota(w) + \iota(w)\iota(v) = 2\langle v, w\rangle \cdot 1_C.$$

$$\text{(so e.g. } \iota(v)^2 = \langle v,v\rangle \cdot 1_C \text{)}$$

This thing exists, is naturally $\mathbb{Z}_2$-graded, $V \hookrightarrow C(V)^1$ is injective and $C(V)$ is $2^{\dim(V)}$ dimensional. Observe that if $T: V \longrightarrow W$ is a morphism in Q that we get a unique morphism of ($\mathbb{Z}_2$-graded) algebras $C(T)$ making the following diagram commute:

$$\begin{tikzcd} V \arrow[r, "T"] \arrow[d] & W \arrow[d] \\ C(V) \arrow[r, dashed, "C(T)"'] & C(W) \end{tikzcd}$$

$$T(v)T(w) + T(w)T(v) = 2\langle Tv, Tw\rangle \cdot 1$$
$$= 2\langle v, w\rangle \cdot 1$$

Lemma $C(-)$ is a strong monoidal functor $Q \longrightarrow Alg_{\mathbb{R}}^{\mathbb{Z}_2}$, i.e. there are natural isomorphisms $C(0) \cong \mathbb{R}$ and

$$C(\underbrace{V \otimes W}_{\text{really direct sum!}}) \cong C(V) \otimes_{\mathbb{R}} C(W).$$

You know these algebras:

$C(X_{0,0}) \cong \mathbb{R}$

$C(X_{0,1}) \cong \mathbb{C}$ (spanned by $1, \frac{\partial}{\partial x_1}$, with $(\frac{\partial}{\partial x_1})^2 = -1$)

$C(X_{0,2}) \cong \mathbb{H}$ (spanned by $1, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}$, the last three of which square to -1 and anticommute)

Remark The functor $C(-)$ is faithful but not full, as e.g. the linear map $\mathbb{R}\frac{\partial}{\partial x_1} \longrightarrow C(X_{0,2})$ sending $\frac{\partial}{\partial x_1} \longmapsto \frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}$ lifts to a morphism of $\mathbb{Z}_2$-graded algebras $C(X_{0,1}) \longrightarrow C(X_{0,2})$ not in the image of $C(-)$.

OK, so there are some interesting things to say about morphisms between these Clifford algebras. But for the truly deep statement we need to go one level up, to categories of modules:

critical point x^* of f $\rightsquigarrow$ quadratic space $(T_{\underline{x}^*}U, H_f|_{\underline{x}^*})$

$\rightsquigarrow$ Clifford algebra $C(T_{\underline{x}^*}U, H_f|_{\underline{x}^*})$

$\rightsquigarrow$ Abelian category $\mathrm{Mod}^{\mathbb{Z}_2} C(T_{\underline{x}^*}U, H_f|_{\underline{x}^*})$

- quadratic space $(T_{\underline{x}^*}U, H_f|_{\underline{x}^*})$ $\qquad V \xrightarrow{T} W$
- Clifford algebra $C(T_{\underline{x}^*}U, H_f|_{\underline{x}^*})$ $\qquad C(V) \xrightarrow{C(T)} C(W)$
- Abelian category $\mathrm{Mod}^{\mathbb{Z}_2} C(T_{\underline{x}^*}U, H_f|_{\underline{x}^*})$.

$$\begin{tikzcd} \mathrm{Mod}^{\mathbb{Z}_2} C(V) \arrow[r, bend left, "C(T)^*"] & \mathrm{Mod}^{\mathbb{Z}_2} C(W) \arrow[l, bend left, "C(T)_*"] \end{tikzcd}$$

<u>Remarkable</u> There are more functors than algebra morphisms! And it is at this level of generality that we find something sufficiently nontrivial to justify talking about morphisms of critical points in the first place.

<u>Theorem</u> (Bott periodicity) There is an equivalence of categories, for all p, q

$$\mathrm{Mod}^{\mathbb{Z}_2} C(X_{p,q+8}) \cong \mathrm{Mod}^{\mathbb{Z}_2} C(X_{p,q})$$

(not induced by an algebra isomorphism)

I do not know an interpretation of this for classical physics, but such periodicity results have been used in the context of $N=2$ supersymmetric conformal field theories and Landau-Ginzburg models, where the content is that theories with these potentials are "the same" (e.g. their boundary conditions and defects with any other theory are equivalent).
(see Vafa-Warner "Catastrophes and the classification of conformal field theories" for the general story of singularities vs. CFT, and Brunner-Hori-Hosomichi-Walcher "Orientifolds of Gepner models" for relevance of periodicity).
So we have (pseudo) functors which are both suitably strongly monoidal

$$\mathcal{Q} \xrightarrow{C(-)} \mathrm{Alg}_{\mathbb{R}}^{\mathbb{Z}_2} \longrightarrow \mathcal{A}lg_{\mathbb{R}}^{\mathbb{Z}_2}$$

quadratic spaces	$\mathbb{Z}_2$-graded algebras (algebra maps)	$\mathbb{Z}_2$-graded algebras bimodules (f.d.) bimodule maps (bicategory)

<u>Def^n</u> The "correct" category of nondegenerate critical points is the symmetric monoidal bicategory $\mathcal{C}rit^{ndg}$ of Clifford algebras $C(V)$ and $\mathbb{Z}_2$-graded f.d. bimodules.

$$\begin{tikzcd} C(V) \arrow[r, bend left, "B"] \arrow[r, bend right, "B'"'] \arrow[r, phantom, "\Downarrow\phi"] & C(W) \end{tikzcd}$$

bimodule

bimodule map

so for each V, W we have a <u>category</u>

$$\mathcal{C}rit^{ndg}(V,W) = \mathrm{Bimod}^{\mathbb{Z}_2}_{f.d.}(C(V), C(W)).$$

a composition <u>functor</u> and a functor of bicategories

$$\otimes : \mathcal{C}rit^{ndg} \times \mathcal{C}rit^{ndg} \longrightarrow \mathcal{C}rit^{ndg}$$
$$(C(V), C(W)) \longmapsto C(V) \dot{\otimes} C(W) \cong C(V \otimes W)$$

satisfying a long list of coherence conditions. (inc. braiding and syllepsis)

<u>Remark</u> In a bicategory $\mathcal{B}$ we write $X \simeq Y$ (i.e. Morita equivalence) if there are $F : X \to Y$, $G : Y \to X$ such that $F \circ G \cong 1_Y$, $G \circ F \cong 1_X$. Then in $\mathcal{C}rit^{ndg}$, Bott-periodicity says

$$X_{0,1}^{\otimes 8} \simeq X_{0,0}, \quad (X_{1,0}^{\otimes 8} \simeq X_{0,0})$$

In fact $\mathcal{C}rit^{ndg}$ is a monoidal bicategory with duals and $X_{1,0}^{\vee} \simeq X_{0,1}$. Really we should write $\mathcal{C}rit^{ndg}_{\mathbb{R}}$, as this monoidal bicategory with duals exists for any (char. 0, let's say) field k, and $\mathcal{C}rit^{ndg}_{\mathbb{C}}$ is generated (using also duals) by an object which is 2-periodic, i.e. $X_{0,1}^{\otimes 2} \simeq X_{0,0}$, this is usually called <u>Knörrer periodicity</u>.

Part III : Degenerate critical points

As we have already discussed, if we care about families of vector fields (e.g. time-varying potentials), and we do, then to understand the dynamical implications of a critical point it is not enough to only consider nondegenerate critical points. In fact the story of general (still isolated) critical points is much more interesting

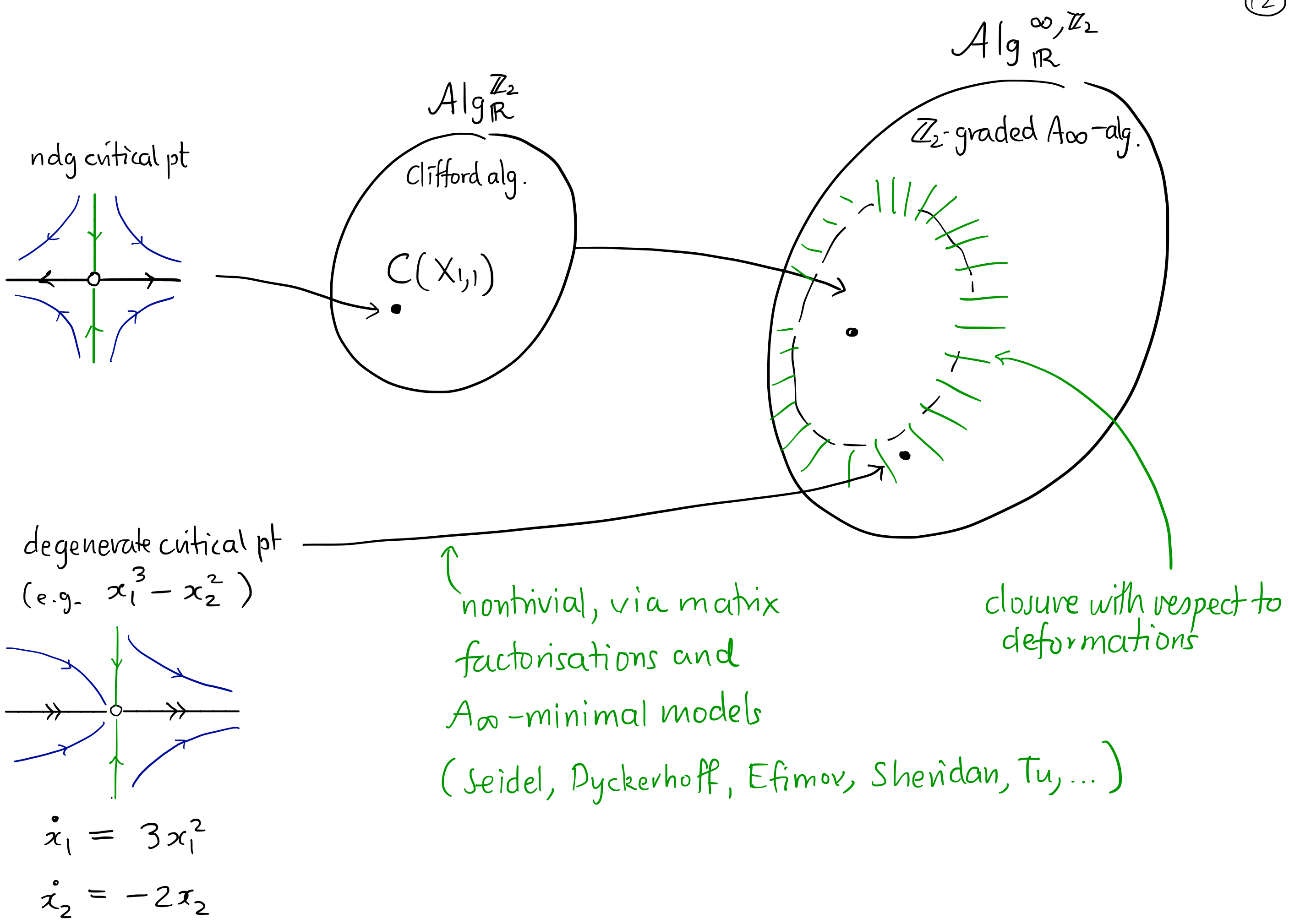
Question : is there a symmetric monoidal bicategory with duals $\mathcal{C}rit_{\mathbb{R}}$ of all isolated critical points and a monoidal functor

$$\mathcal{C}rit^{ndg}_{\mathbb{R}} \xhookrightarrow{\otimes} \mathcal{C}rit_{\mathbb{R}} \ ?$$

The obvious problem is that it is not clear what algebra we should attach to a critical point that is degenerate. Somehow this algebra should know not only about the second derivatives of f (which went into the product in the Clifford algebra) but about all derivatives, but how to put this information in? Here is the idea :

- $C(T_{\underline{x}^*}U, H_f|_{\underline{x}^*})$ was associative but failed to be commutative, with the failure measured by the Hessian $H_f = \left(\partial^2 f / \partial x_i \partial x_j \right)_{i,j}$.

- to a critical point $\underline{x}^*$ of a potential f we associate a $\mathbb{Z}_2$-graded A_∞-algebra $C(f, \underline{x}^*)$, a deformation of the Clifford algebra of the quadratic part of f at $\underline{x}^*$, whose higher products contain the information from the rest of the Taylor expansion of f around $\underline{x}^*$. (via matrix factorisations, see e.g. M "Constructing A_∞-categories of matrix factorisations" for references).



There is a bicategory $\mathcal{C}rit_{\mathbb{R}}$ whose objects are these A_∞-algebras $C(f, \underline{x}^*)$ and whose 1-morphisms are A_∞-bimodules (this is a restatement of a result in N. Carqueville – D.M. "Adjoints and defects in Landau-Ginzburg models"), and more recently it has been shown that:

<u>Theorem</u> (Carqueville – Montoya '18) $\mathcal{C}rit_{\mathbb{R}}$ is a symmetric monoidal bicategory in which every object is fully dualisable, and therefore determines an extended 2D framed TFT

$$\mathrm{Bord}^{fr}_{2,1,0} \longrightarrow \mathcal{C}rit_{\mathbb{R}}.$$

$\mathcal{LG}_{\mathbb{R}}$ in their notation

Moreover $\mathcal{C}rit^{ndg}_{\mathbb{R}} \subset \mathcal{C}rit_{\mathbb{R}}$.

essentially due to Buchweitz – Eisenbud – Herzog.

<u>Remark</u> Given a monoidal bicategory $\mathcal{B}$ and $A \in \mathcal{B}$ a <u>right dual</u> to A is an object A^* and 1-morphisms $ev_A : A \otimes A^* \longrightarrow I$, $coev_A : I \longrightarrow A^* \otimes A$ together with <u>cusp</u> <u>2-isomorphisms</u> (think Zorro moves $\Rightarrow |$, $\Rightarrow |$ but for 2-dimensional regions)
If $\mathcal{B}$ is symmetric monoidal, A is <u>fully dualisable</u> if it has a dual object such that both ev_A, $coev_A$ have both left and right adjoints.

<u>Remark</u> We are eliding the difference between germs of smooth (holomorphic) functions, i.e. convergent <u>powerseries</u>, and <u>polynomials</u> (the def$^{\text{n}}$s of the bicategory $\mathcal{LG}_R$ is algebraic and only allows polynomial potentials). This is harmless because of finite determinacy (see § I.2.2. of Greuel–Lossen–Shustin "Introduction to singularities and deformations").