Introduction.

λ-Calculus

Def^n: Pre-term.

Assume given an infinite set V of variables $x, y \dots$

The set Λ' of pre terms is defined inductively:

- $x \in V \Rightarrow x \in \Lambda'$ ... Variables
- $M, N \in \Lambda' \Rightarrow (MN) \in \Lambda'$ ... Applications
- $x \in V, M \in \Lambda' \Rightarrow (\lambda x. M) \in \Lambda'$ ... Abstractions.

Eg)

Is a pre-term:	Not a pre-term:
x	$()$
xy	λx
$\lambda x. xy$	$\lambda x. ()$
$xy \lambda x. xy$	$x_1 x_2 \dots$

Interpretation of abstractions:

$\lambda x. M: \ x \mapsto M$

$\lambda x. xy: \ x \mapsto xy$

Def^n: Free variables: (let $x \in V, M, N \in \Lambda'$)

The set $FV(\cdot)$ of free variables is defined:

- $FV(x) = \{x\}$
- $FV(MN) = FV(M) \cup FV(N)$
- $FV(\lambda x. M) = FV(M) - \{x\}$.

Def^n: If $y \notin M$, we write $M[x:=y]$ to mean replace x in M with y.

Def^n: α-equivalence: $(\lambda x. M) =_\alpha (\lambda y. M[x:=y])$.

Def^n: The set Λ of λ-terms: $\Lambda = \Lambda' /_{=_\alpha}$

Def^n: $M[x:=N]$ replace x in M with N.
(Only replace free occurences of x)
(Make sure we don't capture a free variable).

eg) $(\lambda y.x)[x:=\lambda z.zy] =_\alpha (\lambda w.x)[x:=\lambda z.zy]$
$= \lambda y.\lambda z.zy \quad \neq_\alpha \quad \lambda w.\lambda z.zy$.

Def^n: β-reduction:
$Q \rightarrow_\beta Q'$ iff Q' is obtained from Q by replacing an occurence of $\underbrace{((\lambda x.M)N)}_{\beta\text{-redex}}$ with $M[x:=N]$)

Eg) $(\lambda x.x)y \rightarrow_\beta y$
$((\lambda x.xy)z)(\lambda z.zz) \rightarrow_\beta (zy)(\lambda z.zz)$
$((\lambda x.x)y)((\lambda x.x)z) \not\rightarrow_\beta yz$
$y(\lambda x.x) \not\rightarrow_\beta y$

Def^n: Multistep β-reduction:
$Q \twoheadrightarrow_\beta Q'$ iff Q' is obtained from Q by a chain of β-reductions, or if $Q = Q'$.

$*$

Let $\lambda x.x = I$
$\lambda x.xx = \omega$

$$\begin{tikzcd}[column sep=small]
\omega(II) \arrow[rrrr, "\beta"'] \arrow[d, "\beta"] & & & & \omega I \arrow[d, "\beta"] \\
(II)(II) \arrow[r, "\beta"] \arrow[dr, "\beta"'] & I(II) \arrow[rrr, "\beta"] & & & II \\
& (II)I \arrow[urrr, "\beta"'] & & &
\end{tikzcd}$$

Church-Rosser Theorem:
if $M_1 \twoheadrightarrow_\beta M_2$, $M_1 \twoheadrightarrow_\beta M_3$, then there exists a λ-term M_4 s.t. $M_2 \twoheadrightarrow_\beta M_4$, and $M_3 \twoheadrightarrow_\beta M_4$.

Direct proof:
If we try induction on $\ell(M_1)$, the length of M_1, we would need: $P_1 \twoheadrightarrow_\beta Q_1$, and $P_2 \twoheadrightarrow_\beta Q_2$, then $P_1[x := P_2] \twoheadrightarrow_\beta Q_1[x := Q_2]$.
To prove this, we need to know the shape of Q_1 from $P_1 \twoheadrightarrow_\beta Q_1$.
Counter example, let $W = \lambda x.xxx$,
can show: $WW \twoheadrightarrow_\beta WWW$
$WW \twoheadrightarrow_\beta WWWW$
$\vdots$

So direct proof won't work.
Try diagram chase. Say we could prove the Church-Rosser th$^{\underline{m}}$ with $\rightarrow_\beta$ replacing all occurrences of $\twoheadrightarrow_\beta$, then we could say:

$$\begin{array}{ccccccc}
M_1 = M_1^{(1)(1)} & \rightarrow_\beta & M_1^{(1)(2)} & \rightarrow_\beta \cdots \rightarrow_\beta & M_1^{(1)(n)} & = M_2 \\
\downarrow_\beta & & \downarrow_\beta & & \downarrow_\beta & \\
M_1^{(2)(1)} & \rightarrow_\beta & M_1^{(2)(2)} & \rightarrow_\beta \cdots \rightarrow_\beta & M_1^{(2)(n)} & \\
\downarrow_\beta & & \downarrow_\beta & & \downarrow_\beta & \\
\vdots & & \vdots & & \vdots & \\
\downarrow_\beta & & \downarrow_\beta & & \downarrow_\beta & \\
M_3 = M_1^{(m)(1)} & \rightarrow_\beta & M_1^{(m)(2)} & \rightarrow_\beta \cdots \rightarrow_\beta & M_1^{(m)(n)} &
\end{array}$$

where blue pen denotes implied existence.

then we could set $M_1^{(m)(n)} = M_4$ and we would be done. But the Church-Rosser th$^{\underline{m}}$ is untrue with $\twoheadrightarrow_\beta$ replaced by $\rightarrow_\beta$, and $*$ is a counter-example.

Def^n: Parallel β-reduction.
Is the least relation on Λ s.t: $(\forall P, P_1, P_2, Q, Q_1, Q_2 \in \Lambda)$

- $x \Rightarrow_\beta x$, $\forall x \in V$ ... ref
- if $P \Rightarrow_\beta Q$, then $\lambda x.P \Rightarrow_\beta \lambda x.Q$ ... lam
- if $P_1 \Rightarrow_\beta Q_1$, and $P_2 \Rightarrow_\beta Q_2$, then $P_1P_2 \Rightarrow_\beta Q_1Q_2$ ... app
- if $P_1 \Rightarrow_\beta Q_1$, and $P_2 \Rightarrow_\beta Q_2$, then $(\lambda x.P_1)P_2 \Rightarrow_\beta Q_1[x:=Q_2]$ ... β_0

Need to prove: $\forall P, P_1, P_2, Q, Q_1, Q_2 \in \Lambda$:

i) if $P \rightarrow_\beta Q$, then $P \Rightarrow_\beta Q$
ii) if $P \Rightarrow_\beta Q$, then $P \twoheadrightarrow_\beta Q$
iii) if $P_1 \Rightarrow_\beta Q_1$, and $P_2 \Rightarrow_\beta Q_2$, then $P_1[x:=P_2] \Rightarrow_\beta Q_1[x:=Q_2]$.
iv) if $P \Rightarrow_\beta P_1$, and $P \Rightarrow_\beta P_2$, then there exists a lambda term P_3 s.t: $P_1 \Rightarrow_\beta P_3$, and $P_2 \Rightarrow_\beta P_3$.

i) and ii) Define: $B_{\rightarrow_\beta} = \{(M,N) \in \Lambda \times \Lambda \mid M \rightarrow_\beta N\}$
$B_{\Rightarrow_\beta} = \{(M,N) \in \Lambda \times \Lambda \mid M \Rightarrow_\beta N\}$
$B_{\twoheadrightarrow_\beta} = \{(M,N) \in \Lambda \times \Lambda \mid M \twoheadrightarrow_\beta N\}$

Then show that $B_{\rightarrow_\beta} \subseteq B_{\Rightarrow_\beta} \subseteq B_{\twoheadrightarrow_\beta}$.

iii) Define: $S_0 = \{(M,M) \mid M \in \Lambda\}$, and; let $(P,Q) \in \Lambda \times \Lambda$, then:
$(P,Q) \in S_{i+1}$ iff:

- $P = \lambda x.P'$, $Q = \lambda x.Q'$, with $(P',Q') \in S_i$
- $P = (P_1'P_2')$, $Q = (Q_1'Q_2')$, with (P_1',Q_1'), $(P_2',Q_2') \in S_i$
- $P = (\lambda x.P_1)P_2$, $Q = Q_1'[x:=Q_2]$, with (P_1',Q_1'), $(P_2',Q_2') \in S_i$
- $(P,Q) \in S_i$ (where $P', Q', P_1', Q_1', P_2', Q_2' \in \Lambda$).

Let $S = \bigcup_i S_i$

Claim: $S = B_{\Rightarrow_\beta}$.

$\Rightarrow_\beta$ is the least relation on λ-terms containing S_0 and satisfying the properties in S, so $B_{\Rightarrow_\beta} \subseteq S$.

To show $B_{\Rightarrow\beta} \subseteq S'$, we prove $B_{\Rightarrow\beta} \subseteq S'_i$ for all i by induction on i.
Know $S'_0 \subseteq B_{\Rightarrow\beta}$
Assume $S'_i \subseteq B_{\Rightarrow\beta}$
Using the axioms of $B_{\Rightarrow\beta}$, we show that $S'_{i+1} \subseteq B_{\Rightarrow\beta}$.
So $S' = B_{\Rightarrow\beta}$ $\square$

Define: $\varphi: S' \longrightarrow \mathbb{N}$
$$(P,Q) \mapsto \min\{i \mid (P,Q) \in S'_i\}$$

Will now prove iii) by induction on $\varphi(P_1, Q_1)$.

Say $\varphi(P_1, Q_1) = 0$, $\Rightarrow P_1 = Q_1$.
Since $P_2 \Rightarrow_\beta Q_2$, we have:
$P_1[x:=P_2] \Rightarrow_\beta P_1[x:=Q_2]$
Now assume the result is true for all $P, Q \in \Lambda$, with $P \Rightarrow_\beta Q$, s.t. $\varphi(P,Q) < k$.
3 cases: P_1 is a variable, abstraction, or application.
Case 1: Reduces to base case.
Case 2: $P_1 = \lambda y.M$, since $P_1 \Rightarrow_\beta P_2$, we know that $P_2 = \lambda y.M'$, with $M \Rightarrow_\beta M'$.
Since $\varphi(M, M') < k$, we have:
$$M[x:=P_2] \Rightarrow_\beta M'[x:=Q_2]$$
$$\Rightarrow \lambda y.M[x:=P_2] \Rightarrow_\beta \lambda y.M'[x:=Q_2], \text{ by lam}$$
$$\Rightarrow (\lambda y.M)[x:=P_2] \Rightarrow_\beta (\lambda y.M')[x:=Q_2]$$
$$\Rightarrow P_1[x:=P_2] \Rightarrow_\beta P_2[x:=Q_2] \quad //$$
Case 3: $P_1 = P_1^{(1)} P_1^{(2)}$, two subcases; either $P_1^{(1)}$ is an abstraction, or it is not.
Say it is not.
Since $P_1 \Rightarrow_\beta Q_1$, then $Q_1 = Q_1^{(1)} Q_1^{(2)}$ with $P_1^{(1)} \Rightarrow_\beta Q_1^{(1)}$, and $P_1^{(2)} \Rightarrow_\beta Q_1^{(2)}$.
So we have:
$$\left.\begin{array}{l} P_1^{(1)}[x:=P_2] \Rightarrow_\beta Q_1^{(1)}[x:=Q_2], \text{ and} \\ P_1^{(2)}[x:=P_2] \Rightarrow_\beta Q_1^{(2)}[x:=Q_2] \end{array}\right\} \text{ by inductive premise.}$$
So we have: $(P_1^{(1)}[x:=P_2])(P_1^{(2)}[x:=P_2]) \Rightarrow_\beta (Q_1^{(1)}[x:=Q_2])(Q_1^{(2)}[x:=Q_2])$ by app
which implies, $(P_1^{(1)} P_1^{(2)})[x:=P_2] \Rightarrow_\beta (Q_1^{(1)} Q_1^{(2)})[x:=Q_2]$
So, $P_1[x:=P_2] \Rightarrow_\beta Q_1[x:=Q_2]$ //

Say $P_1^{(1)}$ is an abstraction; so $P_1^{(1)} P_1^{(2)} = (\lambda x. M) P_1^{(2)}$, some $M \in \Lambda$.
Two sub-subcases, $Q_1 = (\lambda y. M') Q_1^{(2)}$, with $M \Rightarrow_\beta M'$, and $P_1^{(2)} \Rightarrow_\beta Q_1^{(2)}$, in which case, proof is similar to the subcase of case 3 where $P_1^{(1)}$ was assumed not to be an abstraction,
or, $Q_1 = M'[y := P_1^{(3)}]$, with $M \Rightarrow_\beta M'$, and $P_1^{(2)} \Rightarrow_\beta P_1^{(3)}$.
In this case, we have:

$$\left.\begin{array}{l} \cdot\ M[x := P_2] \Rightarrow_\beta M'[x := Q_2] \text{ and,} \\ \cdot\ P_1^{(2)}[x := P_2] \Rightarrow_\beta P_1^{(3)}[x := Q_2] \end{array}\right\} \text{by inductive premise.}$$

we have, $(\lambda y. M[x := P_2]) P_1^{(2)}[x := P_2]$

$$\Rightarrow_\beta (M'[x := Q_2])[y := P_1^{(3)}[x := Q_2]] \text{, by } \beta.$$

$$= (M'[y := P_1^{(3)}])[x := Q_2]$$

lastly, noticing that $(\lambda y. M[x := P_2]) P_1^{(2)}[x := P_2]$

$$= ((\lambda y. M) P_1^{(2)})[x := P_2] \text{, we see that:}$$

$$P_1[x := P_2] \Rightarrow_\beta Q_1[x := Q_2] \ // \qquad \text{II}$$

iv) Proof by induction on $l(P)$, the length of P.
Base case, $l(P)=1$, which means P is a variable.
Take $P_3 = P$, and we are done.
Inductive step, assume if $Q \Rightarrow_\beta Q_1$, and $Q \Rightarrow_\beta Q_2$, with $l(Q) < l(P)$, then $\exists Q_3 \in \Lambda$ s.t. $Q_1 \Rightarrow_\beta Q_3$, and $Q_2 \Rightarrow_\beta Q_3$.
2 cases, P is an abstraction, or an application.
Case 1: Say $P = \lambda x. A$, some $A \in \Lambda$.
Since $P \Rightarrow_\beta P_1$ and $P \Rightarrow_\beta P_2$, we have $P_1 = \lambda x. A_1$ and $P_2 = \lambda x. A_2$ with $A \Rightarrow_\beta A_1$ and $A \Rightarrow_\beta A_2$.
By inductive hypothesis: $\exists A_3 \in \Lambda$ s.t. $A_1 \Rightarrow_\beta A_3$ and $A_2 \Rightarrow_\beta A_3$.
We can take $P_3 = \lambda x. A_3$ and we are done.
Case 2: $P = AB$, $A, B \in \Lambda$. 2 subcases: A is an abstraction, or is not.
Say it is not;
then $P_1 = A_1 B_1$, $P_2 = A_2 B_2$ with $A \Rightarrow_\beta A_1$, $A \Rightarrow_\beta A_2$, $B \Rightarrow_\beta B_1$, $B \Rightarrow_\beta B_2$.
By inductive premise, $\exists A_3, B_3 \in \Lambda$ s.t. $A_1 \Rightarrow_\beta A_3$, $A_2 \Rightarrow_\beta A_3$, $B_1 \Rightarrow_\beta B_3$, $B_2 \Rightarrow_\beta B_3$.
Take $P_3 = A_3 B_3$ and we are done. //
Say A is an abstraction, say $A = \lambda x. A'$, some $A' \in \Lambda$.
3 sub-subcases:

$$\left.\begin{array}{l} 1)\ P_1 = (\lambda x. A'_1)B_1,\ P_2 = (\lambda x. A'_2)B_2 \\ 2)\ P_1 = (\lambda x. A'_1)B_1,\ P_2 = A'_2[x := B_2] \\ 3)\ P_1 = A'_1[x := B_1],\ P_2 = A'_2[x := B_2] \end{array}\right\} \begin{array}{l} \text{with } A' \Rightarrow_\beta A'_1,\ A' \Rightarrow_\beta A'_2, \\ B \Rightarrow_\beta B_1,\ B \Rightarrow_\beta B_2,\ \text{so} \\ \text{by inductive premise, } \exists A'_3, B_3 \in \Lambda \\ \text{s.t. } A'_1 \Rightarrow_\beta A'_3,\ A'_2 \Rightarrow_\beta A'_3,\ B_2 \Rightarrow_\beta B_3,\ B_1 \Rightarrow_\beta B_3 \end{array}$$

In case 1), take $P_3 = (\lambda x. A'_3)B_3$, done.
Case 2) $P_1 \Rightarrow_\beta A_3'[x := B_3]$, by β_0, and
$A_2'[x := B_2] \Rightarrow_\beta A_3'[x := B_3]$, by iii)
So take $P_3 = A_3'[x := B_3]$ done.
Case 3) $A'_1[x := B_1] \Rightarrow_\beta A'_3[x := B_3]$, by iii)
$A'_2[x := B_2] \Rightarrow_\beta A'_3[x := B_3]$, by iii)
So take $P_3 = A'_3[x := B_3]$ $\square$

Now, to prove the Church - Rosser Theorem.

Say $M_1 \twoheadrightarrow_\beta M_2$, and $M_1 \twoheadrightarrow_\beta M_3$.
This can be written as:

$$\begin{array}{ccccccccc}
M_1 = & M_1^{(1)(1)} & \rightarrow_\beta & M_1^{(1)(2)} & \rightarrow_\beta & \cdots & \rightarrow_\beta & M_1^{(1)(n)} & = M_2 \\
& \downarrow_\beta & & & & & & & \\
& M_1^{(2)(1)} & & & & & & & \\
& \downarrow_\beta & & & & & & & \\
& \vdots & & & & & & & \\
& \downarrow_\beta & & & & & & & \\
M_3 = & M_1^{(m)(1)} & & & & & & &
\end{array}$$

By i), this implies:

$$\begin{array}{ccccccc}
M_1^{(1)(1)} & \Rightarrow_\beta & M_1^{(1)(2)} & \Rightarrow_\beta & \cdots & \Rightarrow_\beta & M_1^{(1)(n)} \\
\Downarrow_\beta & & & & & & \\
M_1^{(2)(1)} & & & & & & \\
\Downarrow_\beta & & & & & & \\
\vdots & & & & & & \\
\Downarrow_\beta & & & & & & \\
M_1^{(m)(1)} & & & & & &
\end{array}$$

By iv) we have:

$$\begin{array}{ccccccc}
M_1^{(1)(1)} & \Rightarrow_\beta & M_1^{(1)(2)} & \Rightarrow_\beta & \cdots & \Rightarrow_\beta & M_1^{(1)(n)} \\
\Downarrow_\beta & & \Downarrow_\beta & & & & \Downarrow_\beta \\
M_1^{(2)(1)} & \Rightarrow_\beta & M_1^{(2)(2)} & \Rightarrow_\beta & \cdots & \Rightarrow_\beta & M_1^{(1)(n)} \\
\Downarrow_\beta & & \Downarrow_\beta & & & & \Downarrow_\beta \\
\vdots & & \vdots & & \ddots & & \vdots \\
\Downarrow_\beta & & \Downarrow_\beta & & & & \Downarrow_\beta \\
M_1^{(m)(1)} & \Rightarrow_\beta & M_1^{(m)(2)} & \Rightarrow_\beta & \cdots & \Rightarrow_\beta & M_1^{(m)(n)}
\end{array}$$

where blue pen denotes implied existence.

By ii) we have:

$$\begin{tikzcd}
M_1^{(1)(1)} \arrow[r, two heads, "\beta"'] \arrow[d, two heads, "\beta"] & M_1^{(1)(2)} \arrow[r, two heads, "\beta"'] \arrow[d, two heads, "\beta"] & \cdots \arrow[r, two heads, "\beta"'] & M_1^{(1)(n)} \arrow[d, two heads, "\beta"] \\
M_1^{(2)(1)} \arrow[r, two heads, "\beta"'] \arrow[d, two heads, "\beta"] & M_1^{(2)(2)} \arrow[r, two heads, "\beta"'] \arrow[d, two heads, "\beta"] & \cdots \arrow[r, two heads, "\beta"'] & M_1^{(2)(n)} \arrow[d, two heads, "\beta"] \\
\vdots \arrow[d, two heads, "\beta"] & \vdots \arrow[d, two heads, "\beta"] & \ddots & \vdots \arrow[d, two heads, "\beta"] \\
M_1^{(m)(1)} \arrow[r, two heads, "\beta"'] & M_1^{(m)(2)} \arrow[r, two heads, "\beta"'] & \cdots \arrow[r, two heads, "\beta"'] & M_1^{(m)(n)}
\end{tikzcd}$$

And since $\twoheadrightarrow_\beta$ is transitive, we have $M_1^{(m)(n)}$ such that $M_1^{(1)(n)} \twoheadrightarrow_\beta M_1^{(m)(n)}$, and $M_1^{(m)(1)} \twoheadrightarrow_\beta M_1^{(m)(n)}$, so take M_4 to be $M_1^{(m)(n)}$ and we are done.