A crash course in simplicial homology with coefficients in $\mathbb{Z}_2$.

Aim for this lecture: to define the simplicial homology groups with $\mathbb{Z}_2$ coefficients associated to a triangulable topological space, and give an example.

Historically, simplicial homology came before singular homology, which is what is mostly used today.

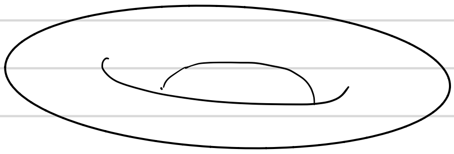
Simplicial homology and singular homology are two different methods for computing (essentially) the same object.

Simplicial homology:
- Positive: More intuitive.
- Negative: Slightly restrictive, is not defined for all spaces.

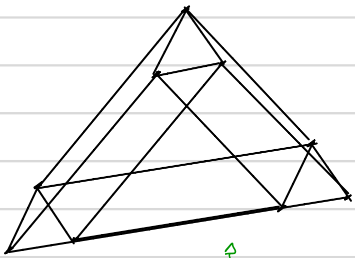
Singular homology:
- Positive: Defined for all spaces.
- ~~Negative~~: Required investment: Pretty much useless without homological algebra.

The idea:

1) Start with a triangulable space X, for example, the torus



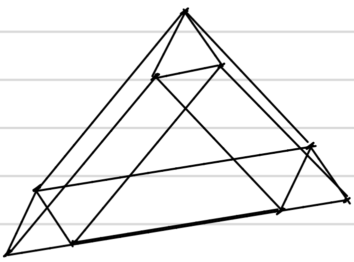
2) Find another space Y, homeomorphic to X, which is "built" by glueing together so called "simplices", which generalise the triangle



A triangulation for each of the rectangular faces is not illustrated but implicitly present.

Such a space Y along with a homeomorphism $h: Y \longrightarrow X$ is called a <u>triangulation of X</u>. If such a pair exists, then X is <u>triangulable</u>.

3) Define an associated sequence of abelian groups,



Some process

Simplicial homology groups of X.

$\rightsquigarrow \quad \{H_n(X)\}_{n=0}^{\infty}$ or

Broad questions:

B1) Why would anyone do this?

Mechanical questions:

M1) What are the definitions involved?

M2) What is an example?

Answers:

B1) • These groups are independent of the triangulation Y.
• These groups provide a topological invariant.

More precisely,

Th$^{\underline{m}}$: Let (Y, h) and (Y', h') both be triangulations for X, and $H_n^Y(X)$ be the nth homology group defined using triangulation (Y, h), then for all n,

$$H_n^Y(X) \cong H_n^{Y'}(X)$$

Isomorphic.

Hence, the Y is dropped from the notation $H_n^Y(X)$.
and,

Th$^{\underline{m}}$: Let X and X' both be triangulable spaces, and assume $X \cong X'$. Then for all n,

$$H_n(X) \cong H_n(X')$$

Homeomorphic.

Isomorphic.

Remark: In fact, more can be said, there is a notion of equivalence of spaces, weaker than homeomorphism, called homotopy equivalence, which also preserves the homology and cohomology groups. This also highlights the important fact that neither homology nor cohomology provide a classification of spaces up to homeomorphism. In fact, neither do either provide a classification for spaces up to homotopy equivalence, but that is another story.

M1) First, we need to be more precise about what a triangulation is. The first step towards this is to generalise the triangle,

Def$^{\underline{n}}$: Let $\{a_0, \dots, a_n\}$ be a set of points in $\mathbb{R}^m$ such that $\{a_1 - a_0, \dots, a_n - a_0\}$ are linearly independent, then the <u>n-simplex</u> spanned by $a_0, \dots, a_n$ is the set

$$\{ x \in \mathbb{R}^m \mid \exists t_0, \dots, t_n \in \mathbb{R}_{\geq 0},\ x = \sum_{i=0}^{n} t_i a_i,\ \sum_{i=0}^{n} t_i = 1 \}$$

A 1-simplex is a line, a 2 simplex is a triangle, 3 simplicies are tetrahedra, etc.

The idea of the hypothesis, is that the boundary of an n-simplex can be given by first connecting n linearly independent points in $\mathbb{R}^n \setminus \{0\}$ to each other, and to the origin, using straight lines, and then translating this whole picture by a_0.

If we demand that n-simplices always have the origin as one of their vertices, then it is clear that the boundary of an n-simplex is to be given by a choice of n linearly independent, non-zero points of $\mathbb{R}^n$. However, we do not want to enforce this restriction, and so any boundary given by a translation of this picture is allowed.

Def^n: A simplicial complex is a set K of simplices such that
- Any simplex spanned by a subset of vertices of any simplex in K is in K,
- The intersection of any two simplicies is a simplex in K.

Def^n: Given a simplicial complex K, let $|K|$ be the union of the simplices of K. This becomes a topological once endowed with the subspace topology.

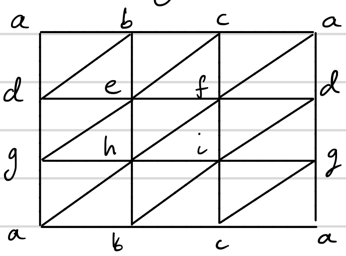
Def^n: A triangulation of a space X is a simplicial complex K along with a homeomorphism $h : |K| \longrightarrow X$. If a triangulation exists, then X is said to be triangulable. The 0-simplicies of K are the verticies of X.

If X is a triangulable space and K is a simplicial complex such that $|K|$ is homeomorphic to X, then if $v_0, \dots, v_n$ are the vertices of X, then the notation $[v_{i_1}, \dots, v_{i_m}]$ will be used to denote the m-simplex with vertices $v_{i_1}, \dots, v_{i_m}$.

Also, to make definitions later on easier to state, it is convenient to assume the vertices are totally ordered.

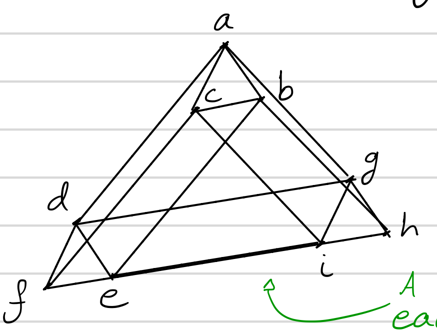
Example:

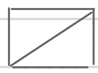
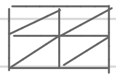
The following is a triangulation of the torus,



(Top and bottom edge identified, left and right edge identified).

which corresponds to the following image, which is truly an embedding in $\mathbb{R}^3$,



Neither  nor  work because the "hole" of the torus made of the curved shapes  and  respectively.

A triangulation for each of the rectangular faces is not illustrated but implicitly present.

For the sake of simplicity, the simplicial homology groups with coefficients in $\mathbb{Z}_2$ will be presented. It should be borne in mind that working with $\mathbb{Z}_2$ coefficients does simplify the story considerably, but is sufficient for the sake of this seminar.

Let X be a triangulable space, and let K be a simplicial complex such that $|K|$ is homeomorphic to X. For each $n \geq 0$, let $C_n(X;\mathbb{Z}_2)$ be the free abelian group with coefficients in $\mathbb{Z}_2$ and basis given by the n-simplices of K. Define the following map for each $n > 0$,

$$\partial_n : C_n(X;\mathbb{Z}_2) \longrightarrow C_{n-1}(X;\mathbb{Z}_2)$$

$$[v_{i_1}, \dots, v_{i_n}] \longmapsto \sum_{j=0}^{n} (-1)^j [v_{i_1}, \dots, \hat{v}_{i_j}, \dots, v_{i_n}] = \sum_{j=0}^{n} [v_{i_1}, \dots, \hat{v}_{i_j}, \dots, v_{i_n}]$$

Omit v_{i_j}.

In $\mathbb{Z}_2$ coefficients.

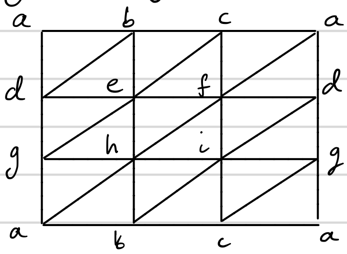
and let $\partial_0 : C_0(X;\mathbb{Z}_2) \longrightarrow 0$ be the zero map.

Thm: $\forall n \geq 0$, $\partial_{n+1}\partial_n = 0$ (Proof is a simple calculation).

Def^n: $H_n(X;\mathbb{Z}_2) = \dfrac{\ker(\partial_n)}{\operatorname{im}(\partial_{n+1})}$ (These are the simplicial homology groups of X with $\mathbb{Z}_2$ coefficients).

<u>Example:</u>

Consider the following triangulation of the torus T,

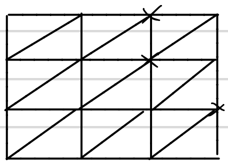


Then by definition, $H_0(T;\mathbb{Z}_2) = \dfrac{C_0(T;\mathbb{Z}_2)}{\operatorname{im}(\partial_1)}$

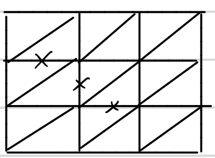
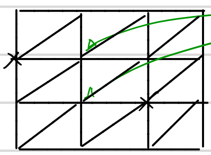
Recall that $C_0(T;\mathbb{Z}_2)$ is the abelian group freely generated by the 0-simplicies of the triangulation, ie, the vertices $a, \dots, i$.

Since the coefficients are in $\mathbb{Z}_2$, an element of $C_0(T;\mathbb{Z}_2)$ is given by marking a subset of the vertices.

"can be thought of as"

Eg, $c + f + g \rightsquigarrow$ 

Similarly, elements of $C_1(T;\mathbb{Z}_2)$ can be thought of as a marked subset of the edges in the triangulation. Let $\sigma \in C_1(T,\mathbb{Z}_2)$, thought of in the above way. Then $\partial_1\sigma$ marks each vertex once for each edge in σ it is an end point of.

Eg  $\xrightarrow{\partial_1}$ 

These edges are marked twice, and thus vanish.

Thus $\operatorname{im}(\partial_1)$ consists of the set of markings of vertices of the triangulation with even cardinality.

Claim: if $\gamma_1, \gamma_2 \in C_0(T,\mathbb{Z}_2)$ both have the same parity of vertices marked (ie, the number of vertices marked are both even, or both odd), then there exists $\sigma \in C_1(T;\mathbb{Z}_2)$ such that $\partial_1\sigma + \gamma_1 = \gamma_2$.

Proof: Wlog, assume the number of vertices marked in γ_1 is less than or equal to the number of vertices marked in γ_2. Let A_i be the set of vertices which are marked by γ_i only (notice that if A_i is empty, then $\gamma_1 = \gamma_2$ and we are done). Then $|A_1| \leq |A_2|$. For each vertex $v \in A_1$, let σ_v be an element of $C_1(T;\mathbb{Z}_2)$ given by a path with one end point at v, and the other at some vertex in A_2 (choose a different vertex in A_2 for each v), this is always possible as T is path connected. Let B be the set of vertices in A_2 corresponding to these paths. Then the cardinality of $A_2 \setminus B$ is necessarily even.
Finally, arbitrarily partition $A_2 \setminus B$ into 2, equal sized subsets, C_1 and C_2. Then for each vertex $u \in C_1$, let σ_u be an element of $C_1(T;\mathbb{Z}_2)$ given by a path from u to some vertex in C_2 (choose a different vertex in C_2 for each $u \in C_1$). Then, define σ to be the sum of all these paths σ_v and σ_u.

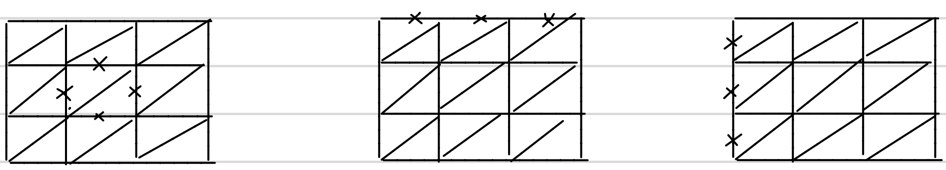
A similar argument shows that if the markings γ_1 and γ_2 are of different parities, then there is no element $\sigma \in C_1(T;\mathbb{Z}_2)$ such that $\gamma_1 + \partial_1\sigma = \gamma_2$.

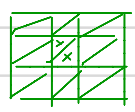
Thus $H_1(T;\mathbb{Z}_2)$ has 2 elements, thus $H_1(T;\mathbb{Z}_2) \cong \mathbb{Z}_2$.

Remark: The above argument holds in more generality than just the torus. In fact it holds for any path connected space. With a little extra work, it can be shown that for a triangulable space X, $H_0(X,\mathbb{Z}_2) \cong \mathbb{Z}_2^k$, where k is the number of path connected components of X. This gives an example of how the <u>algebraic</u> structure of $H_n(X,\mathbb{Z}_2)$ can be used to make statements about the <u>topology</u> of X.

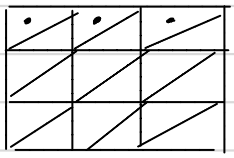
The remaining groups are found in a similar way, the following are some rough sketches which highlight the main ideas.

$H_1(T;\mathbb{Z}_2) = \dfrac{\ker(\partial_1)}{\mathrm{im}(\partial_2)}$ Elements in $\ker(\partial_1)$ correspond to cycles, eg,

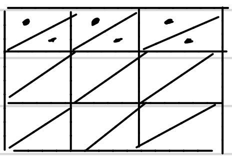


Trivial in $H_1(T;\mathbb{Z}_2)$, as it is $\partial_2\Bigg($  $\Bigg)$

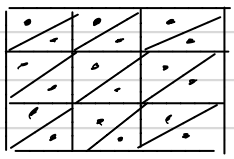
But the other two are non trivial, eg if $\sigma \in C_2(T; \mathbb{Z}_2)$ such that $\partial_2 \sigma$ mapped to the 2^{nd} cycle, then present in σ must be the following marked triangles



But then so must the following,



⋮ etc



But this is a contradiction, as then the edges marked × would be marked twice, and thus vanish.

Similar arguments show that these are the only non-trivial ets, and are distinct. Also, they both have order 2, thus

$$H_1(T; \mathbb{Z}_2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Similarly, $H_2(T; \mathbb{Z}_2) = \ker(\partial_2) \cong \mathbb{Z}_2.$

Final word: As already mentioned, historically, this was the first movement. Now days, people more often use cohomology, rather than homology, which has the benefit of a ring structure, not just an abelian group, leading to a more refined invariant. Again, for the sake of this seminar, what was presented here is sufficient.