The Mock Homotopy Category of Projectives and Grothendieck Duality

Daniel Murfet

September 2007

A thesis submitted, but not yet examined, for the degree of
Doctor of Philosophy of the Australian National University

Declaration

This thesis is an account of research undertaken between March 2004 and September 2007 at the Centre for Mathematics and its Applications, The Australian National University, Canberra, Australia.

Except where acknowledged in the customary manner, the material presented in this thesis is, to the best of my knowledge, original and has not been submitted in whole or part for a degree in any university.

Daniel S. Murfet
September, 2007

Acknowledgements

First and foremost I would like to thank my supervisor Amnon Neeman whose knowledge, support and insight made this project possible. Thanks to Shokrollah Salarian for many helpful conversations about relative homological algebra, and Mark Hovey for answering a question about axiomatic stable homotopy categories.

Abstract

The coherent sheaves defined on a separated noetherian scheme X reflect the underlying geometry, and they play a central role in modern algebraic geometry. Recent results have indicated that there are subtle relationships between projective varieties that are apparent in the properties of bounded *complexes* of coherent sheaves, and so far the most promising way to organize this information is in the bounded derived category of coherent sheaves, which is a triangulated category. There are several other triangulated categories that one can associate to a variety, including the triangulated category of perfect complexes and the triangulated category of singularities.

In this thesis we introduce a compactly generated triangulated category $\mathbb{K}_m(\operatorname{Proj} X)$, called the mock homotopy category of projectives, which extends the derived category of quasi-coherent sheaves by adjoining the acyclic complexes of flat quasi-coherent sheaves. These acyclic complexes carry the same information about the singularities of the scheme as the triangulated category of singularities. Moreover, bounded complexes of coherent sheaves can be viewed as compact objects in the mock homotopy category of projectives, as we establish a duality between the compact objects in this category and the bounded derived category of coherent sheaves on the scheme.

There is another triangulated category, the homotopy category $\mathbb{K}(\operatorname{Inj} X)$ of injective quasi-coherent sheaves, which was introduced earlier by Krause and plays a dual role. In the presence of a dualizing complex we give an equivalence of the mock homotopy category of projectives with the homotopy category of injective quasi-coherent sheaves, interpreting Grothendieck duality as an equivalence of categories of unbounded complexes.

Contents

Chapter 1

Introduction

The philosophy of derived categories is that we should work with complexes rather than their cohomology, which contains less information. This insight continues to be influential, and derived categories now pervade mathematics. As examples of important developments that use the language we quote the duality theory of Grothendieck [Har66], the Riemann-Roch theorem [SGA6], the Riemann-Hilbert correspondence [Kas84], the Kazhdan-Lusztig conjecture [BK81], the Broué conjecture [Bro90], the McKay correspondence [BKR01], the geometric Langlands conjecture [Fre07], homological mirror symmetry [Kon95] and the study of algebraic varieties via their derived categories of coherent sheaves [BO02, Bri06].

However, there is a problem with the derived category, which arises from singularities. Not all invariants must come from cohomology groups: in fact, it has been known for a long time in commutative algebra that there is information about singularities in complexes with no cohomology at all (such complexes are said to be *acyclic*). The relevant invariants occur as *syzygies* (i.e. modules of cocycles) in acyclic complexes. In the derived category it is not possible to talk about such complexes, because we have identified them all with zero. In this thesis we will study a compactly generated triangulated category, which we call the *mock homotopy category of projectives*, that extends the derived category of quasi-coherent sheaves on a scheme by including the acyclic complexes of interest. This builds on earlier work of Krause [Kra05], Jørgensen [Jør05], Iyengar-Krause [IK06] and Neeman [Nee06a].

We begin this introduction with modules over a ring (all our rings are commutative) where it is easier to convince the reader that there is something interesting about acyclic complexes. After explaining the theory in the affine case, which is due to other authors, we state our results which concern the generalization to schemes. Let A be a local Gorenstein ring, and suppose we are given an acyclic complex of finitely generated free A-modules

$$\cdots \longrightarrow L^{-2} \longrightarrow L^{-1} \longrightarrow L^{0} \longrightarrow L^{1} \longrightarrow L^{2} \longrightarrow \cdots$$

The syzygy $M = Ker(L^0 \longrightarrow L^1)$ is such that $\operatorname{depth}(M) = \dim(A)$, and finitely generated modules with this property are known as *maximal Cohen-Macaulay* (MCM) modules. In fact, every MCM module over A occurs in this way, as the syzygy of some acyclic complex of finitely generated free A-modules, called the *complete resolution*; see Lemma 5.11. The category of MCM modules measures the complexity of the singularity of the local ring, and there is a rich algebraic literature on the study of these modules. Let us quote some of the results in the field (suppressing various details):

- Any MCM module over a singularity can be decomposed into a direct sum of indecomposable MCM modules, and a ring has *finite Cohen-Macaulay type* if there are only finitely many indecomposable MCM modules.

- A hypersurface singularity has finite Cohen-Macaulay type if and only if it is a simple singularity; see [Knö87] and [BGS87].

- A surface singularity has finite Cohen-Macaulay type if and only if it is a quotient singularity; this is due independently to Ésnault [Esn85] and Auslander [Aus86].

A lovely survey of rings of finite Cohen-Macaulay type is given in Yoshino's book [Yos90], and see [Dro04, BD06, Ene07] for surveys of more recent work. Having briefly suggested the way in which MCM modules contain information about the nature of singularities, it now remains to argue that the relationship between MCM modules and acyclic complexes is more than a curiousity. This connection is best understood in the context of a body of results now known as *Gorenstein homological algebra*; see [EJ06, Chr00] for background.

Results of Auslander, Auslander-Bridger [AB69] and Auslander-Buchweitz [AB89] on maximal Cohen-Macaulay modules led to the study by Enochs and Jenda of *Gorenstein projective* modules in [EJ95]. A Gorenstein projective module over the local Gorenstein ring A is a module occurring as a syzygy of an acyclic complex of projective A-modules (see [Chr00, §4.2] and [IK06, Corollary 5.5])

$$\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^{0} \longrightarrow P^{1} \longrightarrow P^{2} \longrightarrow \cdots$$

One motivation for introducing Gorenstein projective modules is that the MCM A-modules can now be understood as the *Gorenstein analogues of vector bundles*: a finitely generated A-module is MCM if and only if it is Gorenstein projective (see Lemma 5.11). To formalize the relationship between Gorenstein projectives and acyclic complexes, we introduce two categories:

- The homotopy category $\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ of acyclic complexes of projective A-modules. The objects are the acyclic complexes, and the morphisms are homotopy equivalence classes of cochain maps.

- The stable module category $\operatorname{Gproj}(\underline{\mathbf{Mod}}A)$ of Gorenstein projective A-modules. The objects are the Gorenstein projective A-modules, and the morphisms are morphisms of A-modules modulo the relation that identifies two morphisms if their difference factors via a projective module.

By the definition of Gorenstein projective modules, there is an essentially surjective functor sending an acyclic complex of projective A-modules to its syzygy in degree zero

$$Z^0(-) : \mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A) \longrightarrow \operatorname{Gproj}(\underline{\mathbf{Mod}}A) \tag{1.1}$$

It turns out that this functor, which pairs a module with its so-called *complete projective resolution*, is an equivalence of categories (see Proposition 5.10). Since the MCM modules are precisely the finitely generated Gorenstein projective modules, the equivalence (1.1) clarifies the relationship between MCM modules and acyclic complexes: the stable module category of MCM modules is equivalent to a full subcategory of the homotopy category of acyclic complexes of projective modules. We call the triangulated category $\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ the (*projective*) *stable derived category*; this terminology is explained at length in Krause's paper [Kra05] which had a large influence on how we view the subject.

The information about singularities present in the stable derived category is orthogonal to the information in the ordinary derived category $\mathbb{D}(A)$ of A-modules, where acyclic complexes vanish. To reconcile the two, we need a way of *glueing* the triangulated categories $\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ and $\mathbb{D}(A)$ together; phenomenon such as the Cohen-Macaulay approximation of Auslander and Buchweitz [AB89] suggest that it is fruitful to let the two types of derived category interact. Fortunately, the notion of a *glueing* or *recollement* of two triangulated categories has already been worked out by Beĭlinson, Bernstein and Deligne [BBD82].

The triangulated category that glues $\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ and $\mathbb{D}(A)$ is the homotopy category $\mathbb{K}(\operatorname{Proj} A)$ of projective A-modules, which has arbitrary complexes of projective A-modules as objects, and the homotopy equivalence classes of cochain maps as morphisms. There is a recollement (for any ring, not necessarily local or Gorenstein, see Theorem 5.15)

$$\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A) \leftrightarrows\!\!\rightarrow \mathbb{K}(\operatorname{Proj} A) \leftrightarrows\!\!\rightarrow \mathbb{D}(A) \tag{1.2}$$

in which the six functors describe how to glue the outside objects within the central one. The derived category $\mathbb{D}(A)$ embeds as a subcategory of $\mathbb{K}(\operatorname{Proj} A)$ by identifying a complex with its projective resolution (which makes sense even for unbounded complexes) and as part of the glueing we obtain, for any complex X of projective A-modules, a unique triangle

$$P \longrightarrow X \longrightarrow Z \longrightarrow \Sigma P \tag{1.3}$$

in which P is the projective resolution of an object of $\mathbb{D}(A)$ and Z belongs to $\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A)$. The triangle (1.3) associates X with two kinds of invariants: the cohomology groups of P and the degree zero syzygy of Z, which is a Gorenstein projective module.

The recollement (1.2) gives the desired extension $\mathbb{K}(\operatorname{Proj} A)$ of the derived category by the objects of the projective stable derived category. However, homological algebra is about more than projective resolutions, and the theory we are describing is not complete without its injective aspect: the homotopy category $\mathbb{K}(\operatorname{Inj} A)$ of injective A-modules and the injective stable derived category $\mathbb{K}_{\mathrm{ac}}(\operatorname{Inj} A)$. In what follows we review the results of Krause [Kra05], Jørgensen [Jør05], Iyengar-Krause [IK06] and Neeman [Nee06a] which describe the structure of $\mathbb{K}(\operatorname{Proj} A)$ and $\mathbb{K}(\operatorname{Inj} A)$ in more detail. In a surprising twist, we can deduce from Grothendieck's theory of duality that these two extensions of the derived category are equivalent. First, a brief reminder about compact objects.

1.1. Compact objects in triangulated categories. A triangulated category does not have a lot of structure, so it can be difficult to identify interesting objects without resorting to an underlying model. In a triangulated category $\mathcal{T}$ with infinite coproducts, the most interesting objects are the "finite" ones, known as the *compact* objects. These are objects $x \in \mathcal{T}$ with the property that any morphism from x into an infinite coproduct

$$x \longrightarrow \bigoplus_{i \in I} T_i$$

factors via a finite subcoproduct, indexed by a subset of indices $\{i_0, \dots, i_n\} \subseteq I$

$$x \longrightarrow T_{i_0} \oplus \cdots \oplus T_{i_n} \longrightarrow \bigoplus_{i \in I} T_i$$

We denote the full subcategory of compact objects by $\mathcal{T}^c \subseteq \mathcal{T}$. This notion makes sense for any category with coproducts, and applied to *abelian categories* it identifies the finite objects in the usual sense (for example, finitely generated modules or coherent sheaves).

It is by now a complete triviality to observe that in order to study finitely generated modules, or coherent sheaves, it is worthwhile to study *all* modules, and *all* quasi-coherent sheaves. In many contexts it is appropriate to replace coherent sheaves (and thus abelian categories) by bounded *complexes* of coherent sheaves (and triangulated categories) and it becomes important to know the "infinite completion" which is the triangulated category where such complexes are the compact objects. Moreover, the infinite completion should be *generated* by these compact objects, just as the category of quasi-coherent sheaves is generated by the coherent sheaves (abelian and triangulated categories are quite different, and the sense in which objects generate also differs, see Chapter 2).

The emphasis on compact objects, and compactly generated triangulated categories, was motivated by topology and introduced to the algebraists by Neeman in [Nee92, Nee96]. The main feature of these triangulated categories is that Brown representability theorems and other infinite techniques of homotopy theory are available to study them. There are many applications and we direct the reader to the surveys in [CKN01] and [Nee07].

Given a ring A, the compact objects in the derived category $\mathbb{D}(A)$ are those complexes quasi-isomorphic to a bounded complex of finitely generated projective A-modules. Thus, in the derived category, every "finite" object has finite projective dimension. This seems to be at odds with our intuitive understanding of what a finite complex should be: surely any bounded complex of finitely generated modules

$$0 \longrightarrow M^0 \longrightarrow M^1 \longrightarrow \cdots \longrightarrow M^n \longrightarrow 0 \tag{1.4}$$

deserves to be a "finite" object. If A is a regular noetherian ring of finite Krull dimension then this is actually true in the derived category: complexes of the form (1.4) are compact in $\mathbb{D}(A)$. But in the presence of singularities this no longer holds. In order to make every bounded complex of finitely generated modules a compact object, that is, in order to find the *infinite completion* of the bounded derived category of finitely generated modules, we have to extend the derived category by adjoining acyclic complexes. In other words, we pass to homotopy category $\mathbb{K}(\operatorname{Proj} A)$, or its injective analogue $\mathbb{K}(\operatorname{Inj} A)$.

1.2. The homotopy category of injective modules. Let A be a noetherian ring. The homotopy category $\mathbb{K}(\operatorname{Inj} A)$ of injective A-modules has as objects arbitrary complexes of injective A-modules

$$\cdots \longrightarrow I^{-2} \longrightarrow I^{-1} \longrightarrow I^{0} \longrightarrow I^{1} \longrightarrow I^{2} \longrightarrow \cdots$$

and as morphisms the homotopy equivalence classes of cochain maps. This is a triangulated category, first studied by Krause in his paper [Kra05]. Apart from the homotopy category of injectives, the central object of this paper is the homotopy category $\mathbb{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ of acyclic complexes of injective A-modules, which Krause calls the (*injective*) *stable derived category* of A[1]. The relationship between these two categories is given by a fundamental recollement [Kra05, Corollary 4.3]

$$\begin{tikzcd} \mathbb{K}_{\mathrm{ac}}(\operatorname{Inj} A) \arrow[r] & \mathbb{K}(\operatorname{Inj} A) \arrow[l, shift right=1.5ex] \arrow[l, shift left=1.5ex] \arrow[r] & \mathbb{D}(A) \arrow[l, shift right=1.5ex] \arrow[l, shift left=1.5ex] \end{tikzcd}$$

which glues the derived category $\mathbb{D}(A)$ together with the stable derived category $\mathbb{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ by identifying objects of the derived category with their injective resolution. Adjoining the acyclic complexes of injectives to $\mathbb{D}(A)$ has the effect of making every bounded complex of finitely generated A-modules into a compact object: there is an equivalence of triangulated categories [Kra05, Proposition 2.3]

$$\mathbb{D}^{b}(\operatorname{mod} A) \xrightarrow{\sim} \mathbb{K}^{c}(\operatorname{Inj} A) \tag{1.5}$$

sending a complex on the left to its injective resolution, where $\mathbb{D}^{b}(\operatorname{mod} A)$ is the bounded derived category of finitely generated A-modules. Since $\mathbb{K}(\operatorname{Inj} A)$ is compactly generated it has the necessary properties to play the role of the infinite completion of $\mathbb{D}^{b}(\operatorname{mod} A)$. The stable derived category $\mathbb{K}_{\mathrm{ac}}(\operatorname{Inj} A)$ is compactly generated and contains, in its subcategory of compact objects, the *bounded stable derived category* $\mathbb{D}^{b}_{\mathrm{sg}}(A)$ described by Buchweitz in [Buc87]. There is an equivalence up to direct factors [Kra05, Corollary 5.4]

$$\mathbb{D}^{b}_{\mathrm{sg}}(A) = \mathbb{D}^{b}(\operatorname{mod} A)/\,\mathbb{K}^{b}(\operatorname{proj} A) \xrightarrow{\sim} \mathbb{K}^{c}_{\mathrm{ac}}(\operatorname{Inj} A) \tag{1.6}$$

where $\mathbb{K}^{b}(\operatorname{proj} A)$ is the subcategory of bounded complexes of finitely generated projectives. The quotient in (1.6) describes the additional compact objects that appear in the passage from $\mathbb{D}(A)$ to its extension $\mathbb{K}(\operatorname{Inj} A)$, where the injective resolution of any bounded complex of finitely generated modules is a compact object. There is a second approach to embedding the objects of $\mathbb{D}^{b}(\operatorname{mod} A)$ as compact objects, which goes via projective resolutions.

1.3. The homotopy category of projective modules. Given a noetherian ring A, we have already defined the homotopy category $\mathbb{K}(\operatorname{Proj} A)$ of projective A-modules: it has as objects arbitrary complexes of projective A-modules

$$\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^{0} \longrightarrow P^{1} \longrightarrow P^{2} \longrightarrow \cdots$$

[1]Krause's results are more general, and also apply to schemes; we will discuss the general results shortly.

and as morphisms the homotopy equivalence classes of cochain maps. This is a triangulated category, studied by Jørgensen [Jør05], Iyengar and Krause [IK06] and Neeman [Nee06a]. As discussed above, there is a recollement (Theorem 5.15)

$$\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A) \leftrightarrows \mathbb{K}(\operatorname{Proj} A) \leftrightarrows \mathbb{D}(A)$$

and after adjoining the acyclic complexes of projectives to $\mathbb{D}(A)$ we can identify bounded complexes of finitely generated A-modules with compact objects, but the details are more subtle than in the injective case, because the compact object of $\mathbb{K}(\operatorname{Proj} A)$ corresponding to a finitely generated A-module is *not* its projective resolution (if this were compact, the module would be compact already in the derived category, and thus have finite projective dimension). Instead, there is an equivalence of triangulated categories

$$\mathbb{D}^b(\operatorname{mod} A)^{\mathrm{op}} \xrightarrow{\sim} \mathbb{K}^c(\operatorname{Proj} A) \tag{1.7}$$

sending a bounded complex of finitely generated A-modules to the *dual* $\operatorname{Hom}_A(P, A)$ of its resolution P by finitely generated projectives; see [Jør05, Theorem 3.2] and a related, more general, result of Neeman [Nee06a, Proposition 6.12]. Since $\mathbb{K}(\operatorname{Proj} A)$ is compactly generated (see Theorem 2.30) it is the infinite completion of $\mathbb{D}^b(\operatorname{mod} A)^{\mathrm{op}}$. The stable derived category $\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ follows the same pattern; this category is compactly generated and there is an equivalence up to direct factors [IK06, Theorem 5.3]

$$\mathbb{D}^b_{\mathrm{sg}}(A)^{\mathrm{op}} \xrightarrow{\sim} \mathbb{K}^c_{\mathrm{ac}}(\operatorname{Proj} A) \tag{1.8}$$

In $\mathbb{D}(A)$ there is no difference between a module and its projective or injective resolutions; this is the point of inverting the quasi-isomorphisms. However, in the extensions $\mathbb{K}(\operatorname{Proj} A)$ and $\mathbb{K}(\operatorname{Inj} A)$ of the derived category that we are describing, the projective and injective resolutions of a module exist in completely different categories. The equivalences (1.5), (1.6), (1.7) and (1.8) hint that these categories are related, and in fact the connection is a manifestation of Grothendieck duality.

1.4. Grothendieck duality. Let A be a noetherian ring of finite Krull dimension. The duality theory of Grothendieck describes a special complex D, called the *dualizing complex*, which exists for many rings one encounters in algebraic geometry, including any finitely generated algebra over a field; see [Har66, §II.10] and [Con00, Lemma 3.1.4]. The existence of a dualizing complex for A tells us that the bounded derived category of finitely generated A-modules is self-dual; there is an equivalence

$$\mathbb{R}\operatorname{Hom}_A(-, D) : \mathbb{D}^b(\operatorname{mod} A)^{\mathrm{op}} \xrightarrow{\sim} \mathbb{D}^b(\operatorname{mod} A) \tag{1.9}$$

We recognize the two categories involved in this equivalence as the compacts in $\mathbb{K}(\operatorname{Proj} A)$ and $\mathbb{K}(\operatorname{Inj} A)$, respectively, from (1.5) and (1.7). It is natural to ask if the equivalence of Grothendieck duality extends to an equivalence of the infinite completions, and a theorem of Iyengar and Krause asserts that it does; there is an equivalence [IK06, Theorem 4.2]

$$- \otimes_A D : \mathbb{K}(\operatorname{Proj} A) \xrightarrow{\sim} \mathbb{K}(\operatorname{Inj} A) \tag{1.10}$$

which restricts on compact objects to the equivalence of (1.9). Over a regular ring this is essentially trivial, since $\mathbb{K}(\mathrm{Proj}\, A)$ and $\mathbb{K}(\mathrm{Inj}\, A)$ are both equivalent to the derived category (Section 9.2) but in general it gives a new perspective on the role of the dualizing complex. For example, we can apply $- \otimes_A D$ to acyclic complexes, which are certainly not subject to classical Grothendieck duality; some consequences are discussed in [IK06].

This completes our discussion of the affine case, which involved three main actors: the homotopy category of injectives, the homotopy category of projectives, and Grothendieck duality which relates them. The first of these generalizes immediately to arbitrary schemes, as described by Krause in [Kra05]. In the rest of this introduction we describe our results, which generalize the remaining two.

Setup. For the rest of this introduction, X denotes a **separated noetherian** scheme, and all quasi-coherent sheaves are defined over X.

1.5. The generalization to schemes. In modern algebraic geometry the study of X is largely the study of the coherent sheaves that live on it. We now know that some interesting relationships between varieties only become visible once we enlarge the class of objects under consideration from single coherent sheaves to complexes, and in this connection the bounded derived category $\mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)$ of coherent sheaves[2] becomes a natural object of study; see [BO02, HdB04, Bri06]. A fundamental property of this triangulated category is Grothendieck duality, which describes a special complex $\mathscr{D}$, the *dualizing complex*, with the property that there is an equivalence

$$\mathbb{R}\mathscr{H}om_{\mathrm{qc}}(-, \mathscr{D}) : \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)^{\mathrm{op}} \xrightarrow{\sim} \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X) \tag{1.11}$$

Many schemes admit dualizing complexes, including any variety over a field [Har66, §II.10]. Over a noetherian ring there is an extension of Grothendieck duality (1.9) to the infinite completions (1.10). From the work of Krause we know the infinite completion of the right hand side of (1.11): the homotopy category $\mathbb{K}(\mathrm{Inj}\, X)$ of injective quasi-coherent sheaves[3] is compactly generated, and there is an equivalence [Kra05, Proposition 2.3]

$$\mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X) \xrightarrow{\sim} \mathbb{K}^c(\mathrm{Inj}\, X) \tag{1.12}$$

For an affine scheme $\mathbb{K}(\mathrm{Proj}\, A)$ is the infinite completion of the left hand side of (1.11), but over general schemes there is a gaping hole on the "projective" side of the equivalence, because there is no good notion of a projective quasi-coherent sheaf: to give an example, for a field k the only projective quasi-coherent sheaf over $\mathbb{P}^1(k)$ is the *zero sheaf* [EEGO04, Corollary 2.3]. This is the problem that is solved in this thesis.

[2] There are several equivalent definitions of this category, and the expert can find a comparison of our notation to her favourite definition in Remark 7.1.

[3] In fact, Krause works with $\mathbb{K}(\mathrm{Inj}\, \mathcal{A})$ for an arbitrary locally noetherian abelian category $\mathcal{A}$, so he treats the affine case and the generalization to schemes simultaneously.

We define a triangulated category $\mathbb{K}_m(\operatorname{Proj} X)$, referred to as the *mock homotopy category of projectives* on X, which generalizes the homotopy category of projective modules over a ring. Objects of this category are not complexes of projective quasi-coherent sheaves; rather, they are complexes of *flat* quasi-coherent sheaves. A description of morphisms in this category is a little more subtle, so we delay it until after stating our theorems. The characterization of the compact objects is the first sign that this generalization is correct, as combining Theorem 4.10 and Theorem 7.4 yields:

Theorem I. *The category* $\mathbb{K}_m(\operatorname{Proj} X)$ *is compactly generated and there is an equivalence*

$$\mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco} X)^{op} \xrightarrow{\sim} \mathbb{K}^c_m(\operatorname{Proj} X)$$

When X has enough vector bundles, for example when it is a quasi-projective variety, this equivalence identifies a coherent sheaf $\mathscr{G}$ with the dual complex $\mathscr{H}om(\mathscr{V}, \mathcal{O}_X)$ where $\mathscr{V}$ is a resolution of $\mathscr{G}$ by vector bundles (see Remark 7.5). The next theorem describes the relationship between the mock homotopy category of projectives and the derived category $\mathbb{D}(\mathfrak{Qco} X)$ of quasi-coherent sheaves. The objects of $\mathbb{K}_m(\operatorname{Proj} X)$ are all complexes of flat quasi-coherent sheaves, so it makes sense to define the subcategory $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ of acyclic complexes, which we call the *mock stable derived category* of X. Theorem 5.5 then asserts:

Theorem II. *There is a recollement*

$$\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \leftleftarrows\!\!\!\rightarrow \mathbb{K}_m(\operatorname{Proj} X) \leftleftarrows\!\!\!\rightarrow \mathbb{D}(\mathfrak{Qco} X)$$

We have seen that there is information about singularities in the category $\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A)$. In the special case of a local Gorenstein ring this arises from the fact that MCM modules can be identified with their complete resolutions, which are acyclic complexes of projective modules. This is precisely the information present in the bounded stable derived category, because there is an equivalence up to direct factors (1.8)

$$\mathbb{D}^b_{\mathrm{sg}}(A)^{\mathrm{op}} \xrightarrow{\sim} \mathbb{K}^c_{\mathrm{ac}}(\operatorname{Proj} A)$$

This connection is treated in [Buc87], which has unfortunately never been published. Over a scheme the bounded stable derived category also goes by the name of the *triangulated category of singularities*, as studied by Orlov [Orl04]

$$\mathbb{D}^b_{\mathrm{sg}}(X) = \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco} X)/\operatorname{Perf}(X)$$

where $\operatorname{Perf}(X)$ is the full subcategory of perfect complexes. The properties of singularities reflected in this quotient are also visible in the compact objects of the triangulated category $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$, as Theorem 5.5 and Theorem 7.9 state:

Theorem III. *The category* $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ *is compactly generated, and there is (up to direct factors) an equivalence*

$$\mathbb{D}^b_{\mathrm{sg}}(X)^{op} \xrightarrow{\sim} \mathbb{K}^c_{m,\mathrm{ac}}(\operatorname{Proj} X)$$

In the case where X admits a dualizing complex $\mathscr{D}$ (which we may always assume is a bounded complex of injective quasi-coherent sheaves) Theorem 8.4 gives the extension of Grothendieck duality to the infinite completions:

Theorem IV. *The equivalence of Grothendieck duality*

$$\mathbb{R}\mathscr{H}om_{\mathrm{qc}}(-,\mathscr{D}) : \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)^{op} \xrightarrow{\sim} \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)$$

extends to an equivalence of triangulated categories

$$-\otimes\mathscr{D} : \mathbb{K}_m(\operatorname{Proj} X) \xrightarrow{\sim} \mathbb{K}(\operatorname{Inj} X)$$

Many triangulated categories are closed monoidal categories. For example, the derived category of modules over a ring with the derived tensor product and derived Hom is a closed monoidal category. In Proposition 6.2 we prove that:

Proposition V. *The triangulated category $\mathbb{K}_m(\operatorname{Proj} X)$ is closed symmetric monoidal: it has a tensor product and function object $\mathbb{R}\mathscr{F}lat(-,-)$ compatible with the triangulation.*

From the equivalence of Theorem IV and the closed monoidal structure on $\mathbb{K}_m(\operatorname{Proj} X)$ we obtain a closed monoidal structure on $\mathbb{K}(\operatorname{Inj} X)$, which has the surprising property that the dualizing complex is the unit object of the tensor product (Proposition B.6). Finally, the role of $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ as an invariant of singularities is expressed by Proposition 9.11, which states that passing to an open subset with the same singularities leaves the mock stable derived category unchanged. This restricts on compact objects to a result of Orlov [Orl04, Proposition 1.14] and gives the projective analogue of [Kra05, Corollary 6.10].

Proposition VI. *If $U \subseteq X$ is an open subset containing every singularity of X then the restriction functor is an equivalence*

$$(-)|_U : \mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \xrightarrow{\sim} \mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} U)$$

It follows from this result that $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ vanishes over regular schemes, in which case we deduce from Theorem II that there is a canonical equivalence of $\mathbb{K}_m(\operatorname{Proj} X)$ with the derived category $\mathbb{D}(\mathfrak{Qco}X)$ of quasi-coherent sheaves. In fact, this is an equivalence of closed monoidal categories; the structure described in Proposition V above reduces to the usual derived tensor and Hom over a regular scheme (Remark 9.8).

This completes our description of the major results. Next we give the definition of the mock homotopy category $\mathbb{K}_m(\operatorname{Proj} X)$. The full details can be found in Chapter 3.

1.6. Defining the mock homotopy category of projectives. In the situations where there are not enough projectives one turns to some kind of flat objects, and this is certainly true in algebraic geometry where resolutions by locally free sheaves play a significant role. We define $\mathbb{K}_m(\operatorname{Proj} X)$ by *taking these flat resolutions seriously.* What is it that makes flat resolutions so inferior to projective resolutions? Their main problem is that they fail to be unique in the homotopy category, but fortunately the theory of *Verdier quotients* or *localizations* of triangulated categories gives us a natural way to remedy this defect.

In order to make flat resolutions unique, we have to kill the difference between any two competing resolutions. This difference will take the form of a special type of complex, but before we can describe these "enemy" complexes we need to explain exactly what a flat resolution of a complex is. A complex $\mathscr{F}$ of quasi-coherent sheaves is $\mathbb{K}$-*flat* if $\mathscr{F} \otimes \mathscr{C}$ is acyclic whenever $\mathscr{C}$ is an acyclic complex of quasi-coherent sheaves. Given an arbitrary complex $\mathscr{X}$ of quasi-coherent sheaves, a $\mathbb{K}$-*flat resolution* is a quasi-isomorphism $\mathscr{F} \longrightarrow \mathscr{X}$ with $\mathscr{F}$ a $\mathbb{K}$-flat complex. Let us agree that all our $\mathbb{K}$-flat resolutions are complexes of flat quasi-coherent sheaves (such resolutions always exist, by Corollary 3.22). Now suppose that we have two $\mathbb{K}$-flat resolutions $\mathscr{F}, \mathscr{F}'$ of the same complex $\mathscr{X}$, as in the diagram

$$\begin{tikzcd} \mathscr{F} \arrow[dr] \arrow[dd, dotted, "\varphi"'] & \\ & \mathscr{X} \\ \mathscr{F}' \arrow[ur] & \end{tikzcd} \tag{1.13}$$

If $\mathscr{F}$ were a $\mathbb{K}$-projective resolution (the analogue of a projective resolution for complexes) then by a standard argument we could find a morphism of complexes φ making the above diagram commute, up to homotopy. Suppose, for the sake of argument, that φ exists and connects our two $\mathbb{K}$-flat resolutions via a commutative diagram (1.13) in the homotopy category $\mathbb{K}(\mathfrak{Qco}X)$ of quasi-coherent sheaves. It is clear that φ must be a quasi-isomorphism, and extending to a triangle in the homotopy category, we have

$$\mathscr{F} \xrightarrow{\varphi} \mathscr{F}' \longrightarrow \mathscr{E} \longrightarrow \Sigma\mathscr{F}$$

We deduce that $\mathscr{E}$ is acyclic and $\mathbb{K}$-flat, because $\mathbb{K}$-flatness is stable under mapping cones. Here is our enemy: the difference between two $\mathbb{K}$-flat resolutions of the same object is an acyclic $\mathbb{K}$-flat complex of flat quasi-coherent sheaves. Denoting by $\mathbb{K}(\operatorname{Flat} X)$ the homotopy category of flat quasi-coherent sheaves (the natural home of $\mathbb{K}$-flat resolutions) we set

$$\mathbb{E}(X) = \{\mathscr{E} \in \mathbb{K}(\operatorname{Flat} X) \,|\, \mathscr{E} \text{ is acyclic and } \mathbb{K}\text{-flat}\}$$

and define $\mathbb{K}_m(\operatorname{Proj} X)$ to be the Verdier quotient where such complexes are zero

$$\mathbb{K}_m(\operatorname{Proj} X) = \mathbb{K}(\operatorname{Flat} X)/\mathbb{E}(X)$$

This definition is speculative, because the morphism φ in (1.13) will not exist in general, but it turns out that $\mathbb{K}$-flat resolutions really are unique and functorial in this category; see Remark 5.9. Our simple demand, that flat resolutions should behave like projective resolutions, defines a triangulated category $\mathbb{K}_m(\operatorname{Proj} X)$ with many nice properties.

As often happens in mathematics, this exposition is not historically correct. The affine case was understood first, by Neeman, from a different direction not motivated by solving any problems with flat resolutions. In [Nee06a] and [Nee06c] Neeman studies, for a ring A, the homotopy category $\mathbb{K}(\operatorname{Proj} A)$ as a subcategory of the homotopy category $\mathbb{K}(\operatorname{Flat} A)$ of flat modules, and he discovers many intriguing relationships. Some of these relationships are summarized in a recollement [Nee06c, Remark 3.2]

$$\mathbb{E}(A) \rightleftarrows \mathbb{K}(\mathrm{Flat}\, A) \rightleftarrows \mathbb{K}(\mathrm{Proj}\, A) \tag{1.14}$$

where $\mathbb{E}(A)$ is the triangulated subcategory of complexes $E \in \mathbb{K}(\mathrm{Flat}\, A)$ with the property that $\mathrm{Hom}_A(P, E)$ is acyclic for every complex P of projective modules; in a more compact notation, $\mathbb{E}(A)$ is the orthogonal[4] $\mathbb{K}(\mathrm{Proj}\, A)^{\perp}$. It follows almost immediately from one of Neeman's results [Nee06a, Corollary 8.4] that the objects of $\mathbb{E}(A)$ are precisely the acyclic $\mathbb{K}$-flat complexes; see Proposition 3.4. One consequence of the recollement (1.14) is that the following composite is an equivalence [Nee06a, Remark 1.12]

$$\mathbb{K}(\mathrm{Proj}\, A) \xrightarrow{\mathrm{inc}} \mathbb{K}(\mathrm{Flat}\, A) \xrightarrow{\mathrm{can}} \mathbb{K}(\mathrm{Flat}\, A)/\mathbb{E}(A) \tag{1.15}$$

This equivalence is very interesting, because the right hand side makes no explicit mention of projective modules. The categories $\mathbb{K}(\mathrm{Flat}\, A)$ and $\mathbb{E}(A)$ both generalize to schemes, so this result of Neeman gives us a way to generalize $\mathbb{K}(\mathrm{Proj}\, A)$, and it was the starting point for the research contained in this thesis.

Setting $X = \mathit{Spec}(A)$ in (1.15) gives an equivalence $\mathbb{K}(\mathrm{Proj}\, A) \xrightarrow{\sim} \mathbb{K}_m(\mathrm{Proj}\, X)$, so that the mock homotopy category $\mathbb{K}_m(\mathrm{Proj}\, X)$ reduces to the homotopy category of projective modules over an affine scheme.

1.7. Contents. We begin in Chapter 2 with a review of background material on triangulated categories, resolutions of complexes and relative homological algebra. In Chapter 3 we define the mock homotopy category $\mathbb{K}_m(\mathrm{Proj}\, X)$ of projectives and establish the tools needed to work with it effectively. In particular, we prove that it has small Homs. Chapter 4 contains our proof that this category is compactly generated. In Chapter 5 we study the mock stable derived category $\mathbb{K}_{m,\mathrm{ac}}(\mathrm{Proj}\, X)$ and prove Theorem II. In Chapter 6 we give the closed monoidal structure on $\mathbb{K}_m(\mathrm{Proj}\, X)$, described above in Proposition V, which is immediately applied in Chapter 7 to classify the compact objects in $\mathbb{K}_m(\mathrm{Proj}\, X)$ and $\mathbb{K}_{m,\mathrm{ac}}(\mathrm{Proj}\, X)$. This will complete the proofs of Theorem I and Theorem III. In Chapter 8 we prove Theorem IV, which gives the equivalence $\mathbb{K}_m(\mathrm{Proj}\, X) \cong \mathbb{K}(\mathrm{Inj}\, X)$ extending Grothendieck duality. In Chapter 9 we study two vignettes on the themes of earlier chapters: the analogue of local cohomology for the mock homotopy category of projectives, and Proposition VI, which involves a new characterization of regular schemes in terms of the existence of complexes of flat quasi-coherent sheaves that are not $\mathbb{K}$-flat.

This brings us to the appendices. In Appendix A we prove that the inclusion of the homotopy category $\mathbb{K}(\mathrm{Flat}\, X)$ of flat quasi-coherent sheaves into the homotopy category $\mathbb{K}(\mathfrak{Qco} X)$ of arbitrary quasi-coherent sheaves has a right adjoint. In Appendix B this fact is applied to define a closed monoidal structure on $\mathbb{K}(\mathrm{Flat}\, X)$ and $\mathbb{K}(\mathrm{Inj}\, X)$. Appendix C is more interesting: we communicate a result of Neeman which says that in any triangulated category with coproducts that you are likely to encounter, the existence of tensor

[4]We put the orthogonal $\perp$ on the opposite side to Neeman, see Chapter 2 for our conventions.

products automatically implies the existence of function objects which respect the triangulation. This fact is used in Chapter 6 to give a quick construction of function objects in $\mathbb{K}_m(\operatorname{Proj} X)$ and $\mathbb{D}(\mathfrak{Qco} X)$.

Setup. In this thesis **schemes are all quasi-compact and separated** and **rings are commutative with identity**, unless specified otherwise. The interested reader will note that "separated" can be replaced throughout with "semi-separated".

A note on some foundational issues: we work in a fixed Grothendieck universe $\mathfrak{U}$, so a *class* is a subset of $\mathfrak{U}$ and a *set* is an element of $\mathfrak{U}$. For the sake of the present discussion, anything else is called a *conglomerate*. Apart from the exceptions we are about to name, all categories have a class of objects and between each pair of objects, a set of morphisms.

Exceptions arise by taking the Verdier quotient of two triangulated categories. Such a construction has a class of objects, but the morphisms between a pair of objects may not form a set, or even be small (i.e. bijective to a set). If in such a quotient the conglomerate of morphisms between every pair of objects is small, then we say it has *small Homs* and treat it as a normal category with complete safety. The Verdier quotients arising here are all of this type.

Chapter 2

Background and Notation

In this chapter we review some background material, which is not intended to be read linearly: the reader can skip to Chapter 3 and refer back as needed. We begin with triangulated categories, focusing on recollements, Bousfield localization and standard properties of compactly generated triangulated categories. Then we recall in Section 2.1 and Section 2.2 some properties of homotopy categories, including a review of $\mathbb{K}$-projective, $\mathbb{K}$-injective and $\mathbb{K}$-flat resolutions. In Section 2.3 we translate some theorems of relative homological algebra into the language of homotopy categories, which is the form in which we will use it in the body of the thesis.

For background on general triangulated categories our references are Neeman [Nee01b] and Verdier [Ver96] while for homotopy categories we refer the reader to Weibel [Wei94]. In a triangulated category $\mathcal{T}$ we always denote the suspension functor by Σ.

Adjunctions. Let $F, G : \mathcal{T} \longrightarrow \mathcal{S}$ be triangulated functors. A *trinatural transformation* $\eta : F \longrightarrow G$ is a natural transformation such that for every $X \in \mathcal{T}$ the following diagram commutes in $\mathcal{S}$

$$\begin{tikzcd} F\Sigma(X) \arrow[r, "\sim"] \arrow[d, "\eta_{\Sigma X}"'] & \Sigma F(X) \arrow[d, "\Sigma\eta_X"] \\ G\Sigma(X) \arrow[r, "\sim"'] & \Sigma G(X) \end{tikzcd}$$

Given triangulated functors $F : \mathcal{A} \longrightarrow \mathcal{B}$ and $G : \mathcal{B} \longrightarrow \mathcal{A}$ a *triadjunction* $G \longrightarrow\!\!\!\!\!\dashv F$ is an adjunction between G and F, with G left adjoint to F, in which the unit $\eta : 1 \longrightarrow FG$ and counit $\varepsilon : GF \longrightarrow 1$ are trinatural transformations. In fact, one of these transformations is trinatural if and only if both are. A triangulated functor has a right (left) adjoint if and only if it has a right (left) triadjoint [Nee01b, Lemma 5.3.6]. Between triangulated functors we only ever consider trinatural transformations, and all our adjunctions between triangulated functors are triadjunctions, so we drop the prefix "tri" from the notation. Given a triangulated functor $F : \mathcal{A} \longrightarrow \mathcal{B}$, we write F_λ for the left adjoint and F_ρ for the right adjoint, when they exist.

Localizing subcategories. A triangulated subcategory $\mathcal{S} \subseteq \mathcal{T}$ is *thick* if every direct summand of an object of $\mathcal{S}$ lies in $\mathcal{S}$, *localizing* if it is closed under coproducts in $\mathcal{T}$, and *colocalizing* if it is closed under products. Given a class of objects $\mathcal{C} \subseteq \mathcal{T}$ we denote by $\mathrm{Thick}(\mathcal{C})$ the smallest thick triangulated subcategory of $\mathcal{T}$ containing the objects of $\mathcal{C}$.

Given a triangulated functor $F : \mathcal{T} \longrightarrow \mathcal{S}$ the *kernel* $Ker(F)$ is the thick triangulated subcategory of $\mathcal{T}$ consisting of those $Y \in \mathcal{T}$ with $F(Y) = 0$. The *essential image* $Im(F)$ is the full subcategory of objects in $\mathcal{S}$ isomorphic to $F(Y)$ for some $Y \in \mathcal{T}$. Provided F is full, the essential image $Im(F)$ is a triangulated subcategory of $\mathcal{S}$. A triangulated functor $F : \mathcal{T} \longrightarrow \mathcal{S}$ is an *equivalence up to direct factors* if it is fully faithful and every $X \in \mathcal{S}$ is a direct summand of $F(Y)$ for some $Y \in \mathcal{T}$.

We say that *idempotents split* in a triangulated category $\mathcal{T}$ if every endomorphism $e : X \longrightarrow X$ with $ee = e$ admits a factorisation $e = gf$ for some $f : X \longrightarrow Y, g : Y \longrightarrow X$ with $fg = 1_Y$. If $\mathcal{T}$ has countable coproducts then idempotents automatically split in $\mathcal{T}$ [Nee01b, Proposition 1.6.8].

Lemma 2.1. *Let $\mathcal{T}$ be a triangulated category with countable coproducts, and $\mathcal{S}$ a triangulated subcategory closed under countable coproducts in $\mathcal{T}$. Then $\mathcal{S}$ is thick.*

Proof. Since $\mathcal{S}$ is a triangulated category with countable coproducts, idempotents split in $\mathcal{S}$. Suppose $X \oplus Y \in \mathcal{S}$ for $X, Y \in \mathcal{T}$ and let $u : X \longrightarrow X \oplus Y, p : X \oplus Y \longrightarrow X$ be canonical. Then $\theta = up$ is idempotent in $\mathcal{S}$ and therefore splits; let $g : X \oplus Y \longrightarrow Q, f : Q \longrightarrow X \oplus Y$ be a splitting with $Q \in \mathcal{S}$, so $\theta = fg$ and $gf = 1$. We have $(1 - \theta)f = 0$, so there is $t : Q \longrightarrow X$ with $ut = f$. One checks that t is an isomorphism, so $X \in \mathcal{S}$ as required. $\square$

In light of this lemma, most of the triangulated subcategories that we encounter are automatically thick.

Orthogonals. For a triangulated subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$, the following triangulated subcategories of $\mathcal{T}$ are called the left and right *orthogonals*, respectively

$$\begin{aligned} {}^{\perp}\mathcal{S} &= \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(X, S) = 0 \text{ for all } S \in \mathcal{S}\} \\ \mathcal{S}^{\perp} &= \{X \in \mathcal{T} \mid \mathrm{Hom}_{\mathcal{T}}(S, X) = 0 \text{ for all } S \in \mathcal{S}\} \end{aligned}$$

Both are thick subcategories of $\mathcal{T}$, with $\mathcal{S}^{\perp}$ colocalizing and ${}^{\perp}\mathcal{S}$ localizing. Given a class of objects $\mathcal{C} \subseteq \mathcal{T}$ we write ${}^{\perp}\mathcal{C}$ for ${}^{\perp}\,\mathrm{Thick}(\mathcal{C})$ and $\mathcal{C}^{\perp}$ for $\mathrm{Thick}(\mathcal{C})^{\perp}$. It is clear that $\mathcal{C}^{\perp}$ is the full subcategory of all $X \in \mathcal{T}$ with $\mathrm{Hom}_{\mathcal{T}}(\Sigma^i C, X) = 0$ for every $i \in \mathbb{Z}$ and $C \in \mathcal{C}$, and similarly for ${}^{\perp}\mathcal{C}$.

Verdier sums. Let $\mathcal{T}$ be a triangulated category with triangulated subcategories $\mathcal{S}, \mathcal{Q}$ and denote by $\mathcal{S} \star \mathcal{Q}$ the full subcategory of $\mathcal{T}$ consisting of objects $X \in \mathcal{T}$ that fit into a triangle with $S \in \mathcal{S}$ and $Q \in \mathcal{Q}$

$$S \longrightarrow X \longrightarrow Q \longrightarrow \Sigma S$$

This subcategory is called the *Verdier sum* of $\mathcal{S}$ and $\mathcal{Q}$. If $\mathrm{Hom}_{\mathcal{T}}(S, Q) = 0$ for every pair $S \in \mathcal{S}, Q \in \mathcal{Q}$ then it is an exercise using [BBD82, Proposition 1.1.11] to check that $\mathcal{S} \star \mathcal{Q}$ is a triangulated subcategory of $\mathcal{T}$.

Verdier quotients. Recall from [Ver96] or [Nee01b, Chapter 2] the construction of the *Verdier quotient* of a triangulated category $\mathcal{D}$ by a triangulated subcategory $\mathcal{C}$. We will need a slightly weaker notion: a *weak Verdier quotient* of $\mathcal{D}$ by $\mathcal{C}$ is a triangulated functor $F : \mathcal{D} \longrightarrow \mathcal{T}$ that satisfies $\mathcal{C} \subseteq Ker(F)$ and is "weakly" universal with this property, in the sense that given any triangulated functor $G : \mathcal{D} \longrightarrow \mathcal{S}$ with $\mathcal{C} \subseteq Ker(G)$ there exists a triangulated functor $H : \mathcal{T} \longrightarrow \mathcal{S}$ together with a natural equivalence $HF \cong G$. Moreover, we require that any two factorizations H, H' with this property be naturally equivalent.

A triangulated functor $F : \mathcal{D} \longrightarrow \mathcal{T}$ is a weak Verdier quotient of $\mathcal{D}$ by $\mathcal{C}$ if and only if it factors as the Verdier quotient $\mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ followed by an equivalence of triangulated categories $\mathcal{D}/\mathcal{C} \xrightarrow{\sim} \mathcal{T}$. We have the following properties: let $F : \mathcal{D} \longrightarrow \mathcal{T}$ be a weak Verdier quotient of $\mathcal{D}$ by $\mathcal{C}$. Then

(i) Given $X, Y \in \mathcal{D}$ the canonical map $\mathrm{Hom}_{\mathcal{D}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{T}}(FX, FY)$ is an isomorphism if either $X \in {}^{\perp}\mathcal{C}$ or $Y \in \mathcal{C}^{\perp}$.

(ii) Given triangulated functors $H, H' : \mathcal{T} \longrightarrow \mathcal{S}$ and a natural transformation $\Phi : HF \longrightarrow H'F$, there is a unique natural transformation $\phi : H \longrightarrow H'$ with $\phi F = \Phi$.

(iii) Suppose we have a diagram of triangulated functors $\mathcal{D} \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{Q}$. If GF has a right adjoint H then G has right adjoint FH.

For proofs of these statements see [Nee01b, Lemma 9.1.5] and [AJS00, Lemma 5.5].

The notion of a *recollement* or *glueing* of triangulated categories was introduced by Beilinson, Bernstein and Deligne in their influential paper [BBD82, §1.4]. In our study of homotopy categories, (co)localization sequences and recollements will provide a powerful organizing principle.

Recollements. We often encounter pairs of functors that, up to equivalence, are the inclusion of, and Verdier quotient by, a triangulated subcategory. This situation is axiomatized as follows: a sequence of triangulated functors

$$\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}'' \tag{2.1}$$

is an *quotient sequence* if the following holds

(E1) The functor F is fully faithful.

(E2) The functor G is a weak Verdier quotient.

(E3) There is an equality of triangulated subcategories $Im(F) = Ker(G)$.

In this case G is a weak Verdier quotient of $\mathcal{T}$ by $Im(F)$. The sequence (2.1) is a quotient sequence if and only if $(\mathcal{T}')^{\mathrm{op}} \longrightarrow \mathcal{T}^{\mathrm{op}} \longrightarrow (\mathcal{T}'')^{\mathrm{op}}$ is a quotient sequence. We say that the sequence of triangulated functors (2.1) is a *localization sequence* if

(L1) The functor F is fully faithful and has a right adjoint.

(L2) The functor G has a fully faithful right adjoint.

(L3) There is an equality of triangulated subcategories $Im(F) = Ker(G)$.

On the other hand, we say that (2.1) is a *colocalization sequence* if the pair $(F^{\mathrm{op}}, G^{\mathrm{op}})$ of opposite functors is a localization sequence, which is equivalent to replacing "right" by "left" in (L1) and (L2). A sequence of functors is a *recollement* if it is both a localization sequence and a colocalization sequence. In this case, the various adjoints are often arranged in a diagram of the following form:

$$\begin{tikzcd} \mathcal{T}' \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] & \mathcal{T} \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] & \mathcal{T}'' \end{tikzcd}$$

The original reference for localization sequences is Verdier's thesis; see [Ver96, §II.2] and [Ver77, §I.2 no.6]. The results were rediscovered by Bousfield [Bou79] and the reader can find more recent expositions in [AJS00, §1], [Nee01b, §9.2] and [Kra05, §3]. The next easy lemma tells us that any (co)localization sequence is a quotient sequence; in particular we have an equivalence $\mathcal{T}/\mathcal{T}' \xrightarrow{\sim} \mathcal{T}''$.

Lemma 2.2. *Every localization or colocalization sequence is a quotient sequence.*

Proof. It is enough to show that every localization sequence (2.1) is a quotient sequence. We prove that the induced functor $M : \mathcal{T}/Im(F) \longrightarrow \mathcal{T}''$ is an equivalence. Let G_ρ be the right adjoint of G with unit $\eta : 1 \longrightarrow G_\rho G$ and set $N = Q \circ G_\rho$ where $Q : \mathcal{T} \longrightarrow \mathcal{T}/Im(F)$ is the Verdier quotient. Then $MN = MQG_\rho = GG_\rho \cong 1$ and it is not difficult to check that $Q\eta : Q \longrightarrow QG_\rho G$ is a natural equivalence. Therefore $NMQ = NG = QG_\rho G \cong Q$ from which we deduce that $NM \cong 1$, as claimed. □

The following lemma gives a useful list of characterizations of localization sequences.

Lemma 2.3. *Suppose we have a quotient sequence of triangulated functors*

$$\mathcal{T}' \xrightarrow{F} \mathcal{T} \xrightarrow{G} \mathcal{T}'' \tag{2.2}$$

The following are equivalent:

(i) The sequence (2.2) is a localization sequence.

(ii) F has a right adjoint.

(iii) G has a right adjoint.

(iv) The composite $Im(F)^\perp \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}''$ is an equivalence.

(v) For every $X \in \mathcal{T}$ there is a triangle

$$L \longrightarrow X \longrightarrow R \longrightarrow \Sigma L$$

with $L \in Im(F)$ and $R \in Im(F)^\perp$.

Proof. Up to equivalence a quotient sequence is of the form $\mathcal{S} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{S}$ for some triangulated subcategory $\mathcal{S}$, so $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ is [Kra05, Lemma 3.2]. For $(v) \Leftrightarrow (iv) \Leftrightarrow (i)$ see [AJS00, Proposition 1.6]. $\square$

Remark 2.4. The dual of Lemma 2.3 characterizes colocalization sequences. To be precise, we replace "right" by "left" in (ii) and (iii), replace $Im(F)^{\perp}$ by ${}^{\perp}Im(F)$ in (iv), and replace (v) by the condition that for every $X \in \mathcal{T}$ there is a triangle $L \longrightarrow X \longrightarrow R \longrightarrow \Sigma L$ with $L \in {}^{\perp}Im(F)$ and $R \in Im(F)$.

Remark 2.5. In a quotient sequence $\mathcal{C} \longrightarrow \mathcal{D} \longrightarrow \mathcal{D}/\mathcal{C}$ the Verdier quotient $\mathcal{D}/\mathcal{C}$ may not have small Homs. But if the quotient sequence is a (co)localization sequence, then $\mathcal{D}/\mathcal{C}$ is equivalent to a subcategory of $\mathcal{D}$, and thus has small Homs if $\mathcal{D}$ does.

We say that a triangulated functor $G : \mathcal{T} \longrightarrow \mathcal{S}$ *induces a localization sequence* (resp. *colocalization sequence, recollement*) if the pair $Ker(G) \longrightarrow \mathcal{T} \longrightarrow \mathcal{S}$ is a localization sequence (resp. colocalization sequence, recollement).

Lemma 2.6. *A triangulated functor $G : \mathcal{T} \longrightarrow \mathcal{S}$ with a fully faithful right adjoint (resp. fully faithful left adjoint) induces a localization sequence (resp. colocalization sequence).*

Proof. We only prove the statement about localization sequences, as the statement about colocalization sequences is dual. Given $X \in \mathcal{T}$ extend the unit morphism $X \longrightarrow G_\rho G(X)$ to a triangle in $\mathcal{T}$

$$Y \longrightarrow X \longrightarrow G_\rho G(X) \longrightarrow \Sigma Y$$

The counit of adjunction is a natural equivalence because G_ρ is fully faithful, so applying G to the triangle we infer that $G(Y) = 0$ (using $\varepsilon G \circ G\eta = 1$). That is, $Y \in Ker(G)$. One checks that $G_\rho G(X) \in Ker(G)^{\perp}$ so the inclusion $Ker(G) \longrightarrow \mathcal{T}$ has a right adjoint [AJS00, Proposition 1.6]. Now, by definition, we have a localization sequence. $\square$

Compactness. Let $\mathcal{T}$ be a triangulated category with coproducts. An object $C \in \mathcal{T}$ is said to be *compact* if every morphism $C \longrightarrow \bigoplus_{i \in I} X_i$ to a coproduct in $\mathcal{T}$ factors through a finite subcoproduct

$$C \longrightarrow X_{i_0} \oplus \cdots \oplus X_{i_n} \longrightarrow \bigoplus_{i \in I} X_i$$

The triangulated category $\mathcal{T}$ is *compactly generated* if there is a set Q of compact objects with the property that any nonzero X in $\mathcal{T}$ admits a nonzero morphism $q \longrightarrow X$ from some $q \in Q$. In this case Q is called a *compact generating set* for $\mathcal{T}$. For any triangulated category $\mathcal{T}$ we write $\mathcal{T}^c \subseteq \mathcal{T}$ for the thick subcategory of compact objects; if $\mathcal{T}$ is compactly generated with compact generating set Q then $\mathcal{T}^c$ is the smallest thick subcategory of $\mathcal{T}$ containing Q and a localizing subcategory of $\mathcal{T}$ containing Q must be all of $\mathcal{T}$. For proofs of these statements see [Nee01b] or [Kra02].

Many deep questions about triangulated categories involve the existence of adjoints, so the next result explains the relevance of compactly generated triangulated categories.

Proposition 2.7. *Let $F : \mathcal{T} \longrightarrow \mathcal{Q}$ be a triangulated functor with $\mathcal{T}$ compactly generated.*

(i) F has a right adjoint if and only if it preserves coproducts.

(ii) F has a left adjoint if and only if it preserves products.

Proof. See [Nee96, Theorem 4.1], [Nee01b, Theorem 8.6.1] and [Kra02]. Note that it is crucial that $\mathcal{T}, \mathcal{Q}$ have small Homs; this result will not necessarily hold if $\mathcal{Q}$ is a "category" where the morphisms between pairs of objects do not form a set. □

Let $\mathcal{T}$ be a triangulated category with coproducts. A localizing subcategory $\mathcal{S} \subseteq \mathcal{T}$ is *compactly generated in* $\mathcal{T}$ if $\mathcal{S}$ admits a compact generating set consisting of objects compact in the larger category $\mathcal{T}$. In this case the inclusion $\mathcal{S} \longrightarrow \mathcal{T}$ has a right adjoint by Proposition 2.7, so there is a localization sequence $\mathcal{S} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{S}$.

Theorem 2.8. *Let $\mathcal{T}$ be a compactly generated triangulated category and $\mathcal{S}$ a localizing subcategory compactly generated in $\mathcal{T}$. Then $\mathcal{S}^c = \mathcal{S} \cap \mathcal{T}^c$ and $\mathcal{T}/\mathcal{S}$ is compactly generated (with small Homs). Moreover, there is an equivalence up to direct factors*

$$\mathcal{T}^c/\mathcal{S}^c \xrightarrow{\sim} (\mathcal{T}/\mathcal{S})^c \tag{2.3}$$

Proof. This is the Neeman-Ravenel-Thomason localization theorem; for the history of this result see [Nee06b]. We apply [Nee92] and [Nee01b, Chapter 4] to deduce that the compact objects in $\mathcal{S}$ are precisely the compact objects of $\mathcal{T}$ that happen to lie in $\mathcal{S}$, that the Verdier quotient $Q : \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{S}$ preserves compactness, and that the canonical functor (2.3) is an equivalence up to direct factors. Observe that by Lemma 2.3 the pair $\mathcal{S} \longrightarrow \mathcal{T} \longrightarrow \mathcal{T}/\mathcal{S}$ is a localization sequence and in particular the right adjoint $Q_\rho : \mathcal{T}/\mathcal{S} \longrightarrow \mathcal{T}$ is fully faithful. This shows that $\mathcal{T}/\mathcal{S}$ has small Homs. Finally, it is not difficult to check that Q sends a compact generating set for $\mathcal{T}$ to a compact generating set for $\mathcal{T}/\mathcal{S}$, which proves that the latter category is compactly generated. □

Lemma 2.9. *Let $F : \mathcal{T} \longrightarrow \mathcal{S}$ be a triangulated functor with right adjoint G. Then*

(i) If G preserves coproducts then F preserves compactness.

(ii) If $\mathcal{T}$ is compactly generated and F sends compact objects to compact objects, then G preserves coproducts.

Proof. See [Nee96, Theorem 5.1]. □

Corollary 2.10. *Suppose that we have a recollement with $\mathcal{T}$ compactly generated*

$$\begin{tikzcd} \mathcal{S} \arrow[r] & \mathcal{T} \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r] & \mathcal{Q} \arrow[l, shift right=2] \arrow[l, shift left=2] \end{tikzcd} \tag{2.4}$$

Then $\mathcal{S}$ is compactly generated, and provided $\mathcal{Q}$ is also compactly generated there is an equivalence up to direct factors $\mathcal{T}^c/\mathcal{Q}^c \xrightarrow{\sim} \mathcal{S}^c$.

Proof. Let $F : \mathcal{S} \longrightarrow \mathcal{T}$ and $G : \mathcal{T} \longrightarrow \mathcal{Q}$ be the pair of triangulated functors that forms the recollement, and denote the left adjoints of F and G by F_λ and G_λ respectively. There is a localization sequence

$$\mathcal{Q} \xrightarrow{G_\lambda} \mathcal{T} \xrightarrow{F_\lambda} \mathcal{S}$$

Since G_λ, F_λ both have coproduct preserving right adjoints it follows from Lemma 2.9 that they preserve compactness. It is not difficult to check that F_λ sends a compact generating set for $\mathcal{T}$ to a compact generating set for $\mathcal{S}$, which is therefore compactly generated. Now assume that $\mathcal{Q}$ is compactly generated. If we identify $\mathcal{Q}$ with a triangulated subcategory of $\mathcal{T}$ via the fully faithful functor G_λ, then $\mathcal{Q}$ is compactly generated in $\mathcal{T}$. From Theorem 2.8 and the equivalence $\mathcal{T}/\mathcal{Q} \xrightarrow{\sim} \mathcal{S}$ induced by the weak Verdier quotient $F_\lambda : \mathcal{T} \longrightarrow \mathcal{S}$ we obtain an equivalence up to direct factors $\mathcal{T}^c/\mathcal{Q}^c \longrightarrow \mathcal{S}^c$, where we identify $\mathcal{Q}^c$ with a subcategory of $\mathcal{T}^c$ via G_λ. □

When triangulated categories $\mathcal{T}$ and $\mathcal{S}$ are equivalent, the full subcategories of compact objects are equivalent. The next result proves a kind of converse, provided we know that the equivalence on compacts lifts to a functor on the whole category.

Proposition 2.11. *Let $F : \mathcal{T} \longrightarrow \mathcal{S}$ be a coproduct preserving triangulated functor between compactly generated triangulated categories. Provided F preserves compactness, it is an equivalence if and only if the induced functor $F^c : \mathcal{T}^c \longrightarrow \mathcal{S}^c$ is an equivalence.*

Proof. This is straightforward to check; see for example [Miy07, Proposition 6]. □

Bousfield subcategories. A *Bousfield subcategory* $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is a thick subcategory $\mathcal{S} \subseteq \mathcal{T}$ with the property that the inclusion $\mathcal{S} \longrightarrow \mathcal{T}$ has a right adjoint (a Bousfield subcategory is automatically localizing). By Lemma 2.3 this is the same as a thick subcategory $\mathcal{S}$ that admits for each $X \in \mathcal{T}$ a triangle with $L \in \mathcal{S}$ and $R \in \mathcal{S}^\perp$

$$L \longrightarrow X \longrightarrow R \longrightarrow \Sigma L \tag{2.5}$$

We note that triangles of this form are unique up to isomorphism; see [BBD82, Proposition 1.1.9]. There is an equivalence $\mathcal{S}^\perp \xrightarrow{\sim} \mathcal{T}/\mathcal{S}$, so the quotient has small Homs.

Rouquier introduced in [Rou03, (5.3.3)] the concept of a *cocovering* of $\mathcal{T}$ by a family of Bousfield subcategories $\{\mathcal{S}_0, \dots, \mathcal{S}_d\}$. Thinking of the subcategories $\mathcal{S}_i$ as closed subsets and the quotients $\mathcal{T}/\mathcal{S}_i$ as open subsets, he shows how to prove statements about $\mathcal{T}$ by arguing over each element $\mathcal{T}/\mathcal{S}_i$ of the "open cover". This idea will be crucial in the proof of one of our major theorems in Chapter 4, so we give the definitions here in some detail.

Elements of a cocovering (defined below) are required to satisfy a technical condition that is automatically satisfied in most examples. Let $\mathcal{T}$ be a triangulated category and let $\mathcal{I}_1, \mathcal{I}_2 \subseteq \mathcal{T}$ be Bousfield subcategories. We say that $\mathcal{I}_1$ and $\mathcal{I}_2$ *intersect properly* if for every pair of objects $M_1 \in \mathcal{I}_1$ and $M_2 \in \mathcal{I}_2$ any morphism in $\mathcal{T}$ of the form

$$M_1 \longrightarrow M_2 \quad \text{or} \quad M_2 \longrightarrow M_1$$

factors through an object of the intersection $\mathcal{I}_1 \cap \mathcal{I}_2$. Given a triangulated category $\mathcal{T}$ and Bousfield subcategories $\mathcal{I}_1, \mathcal{I}_2$ we follow the notation of Rouquier and denote the inclusions by $i_{1*}:\mathcal{I}_1 \longrightarrow \mathcal{T}$ and $i_{2*}:\mathcal{I}_2 \longrightarrow \mathcal{T}$ with right adjoints $i_1^!, i_2^!$. Let $j_1^*:\mathcal{T}\longrightarrow \mathcal{T}/\mathcal{I}_1$ and $j_2^*:\mathcal{T}\longrightarrow \mathcal{T}/\mathcal{I}_2$ be the Verdier quotients with right adjoints j_{1*}, j_{2*}.

Lemma 2.12. *Let $\mathcal{T}$ be a triangulated category with Bousfield subcategories $\mathcal{I}_1,\mathcal{I}_2$. The following conditions are equivalent:*

(i) The subcategories $\mathcal{I}_1$ and $\mathcal{I}_2$ intersect properly.

(ii) $i_{1*}i_1^!(\mathcal{I}_2)\subseteq \mathcal{I}_2$ and $i_{2*}i_2^!(\mathcal{I}_1)\subseteq \mathcal{I}_1$.

(iii) $j_{1*}j_1^*(\mathcal{I}_2)\subseteq \mathcal{I}_2$ and $j_{2*}j_2^*(\mathcal{I}_1)\subseteq \mathcal{I}_1$.

Proof. See [Rou03, Lemma 5.7]. □

Let $\mathcal{T}$ be a triangulated category with coproducts. A *cocovering* of $\mathcal{T}$ is a nonempty finite set $\mathcal{F}=\{\mathcal{T}_0,\dots,\mathcal{T}_d\}$ of Bousfield subcategories of $\mathcal{T}$ with the property that any pair $\mathcal{T}_i,\mathcal{T}_j$ of objects in $\mathcal{F}$ intersect properly and $\mathcal{T}_0\cap\cdots\cap\mathcal{T}_d$ contains only zero objects.

Theorem 2.13. *Let $\mathcal{T}$ be a triangulated category with coproducts and a cocovering $\mathcal{F}$ by Bousfield subcategories. Assume that for all $\mathcal{I}\in\mathcal{F}$ and $\mathcal{F}'\subseteq\mathcal{F}\setminus\{\mathcal{I}\}$ the quotient*

$$\Big(\bigcap_{\mathcal{I}'\in\mathcal{F}'}\mathcal{I}'\Big)\Big/\Big(\bigcap_{\mathcal{I}'\in\mathcal{F}'\cup\{\mathcal{I}\}}\mathcal{I}'\Big)$$

is compactly generated in $\mathcal{T}/\mathcal{I}$. Then $\mathcal{T}$ is compactly generated and $X\in\mathcal{T}$ is compact if and only if it is compact in $\mathcal{T}/\mathcal{I}$ for all $\mathcal{I}\in\mathcal{F}$. Moreover, if $\mathcal{J}$ is a Bousfield subcategory of $\mathcal{T}$ intersecting properly every element of $\mathcal{F}$ with the property that for every $\mathcal{I}\in\mathcal{F}$ and $\mathcal{F}'\subseteq\mathcal{F}\setminus\{\mathcal{I}\}$ the subcategory

$$\Big(\mathcal{J}\cap\bigcap_{\mathcal{I}'\in\mathcal{F}'}\mathcal{I}'\Big)\Big/\Big(\mathcal{J}\cap\bigcap_{\mathcal{I}'\in\mathcal{F}'\cup\{\mathcal{I}\}}\mathcal{I}'\Big)$$

is compactly generated in $\mathcal{T}/\mathcal{I}$, then $\mathcal{J}$ is compactly generated in $\mathcal{T}$.

Proof. See [Rou03, Theorem 5.15]. Note that the subset $\mathcal{F}'$ is allowed to be empty, so in particular the quotients $\mathcal{T}/\mathcal{I}$ must be compactly generated for $\mathcal{I}\in\mathcal{F}$. □

2.1 Homotopy Categories

The triangulated categories of interest to us are homotopy categories and their quotients. Let $\mathcal{X}$ be an additive category and denote by $\mathbb{C}(\mathcal{X})$ the category of all complexes in $\mathcal{X}$. Complexes are usually written cohomologically, as in the following diagram

$$\cdots \longrightarrow X^n \xrightarrow{\partial^n} X^{n+1} \xrightarrow{\partial^{n+1}} X^{n+2} \longrightarrow \cdots$$

The homotopy category $\mathbb{K}(\mathcal{X})$ has as objects the complexes in $\mathcal{X}$ and as morphisms the homotopy equivalence classes of morphisms of complexes. Given an abelian category $\mathcal{A}$ we denote by $\mathbb{K}_{\mathrm{ac}}(\mathcal{A})$ the triangulated subcategory of $\mathbb{K}(\mathcal{A})$ consisting of the complexes X with $H^n(X) = 0$ for all $n \in \mathbb{Z}$. Such complexes are called *exact* or *acyclic*. The (unbounded) derived category $\mathbb{D}(\mathcal{A})$ is defined to be the Verdier quotient of $\mathbb{K}(\mathcal{A})$ by the triangulated subcategory $\mathbb{K}_{\mathrm{ac}}(\mathcal{A})$, with the quotient usually denoted by $q : \mathbb{K}(\mathcal{A}) \longrightarrow \mathbb{D}(\mathcal{A})$.

Let A be a ring and denote by $\mathbf{Mod}A$ the abelian category of A-modules. We write $\mathbb{K}(A)$ for $\mathbb{K}(\mathbf{Mod}A)$ and $\mathbb{D}(A)$ for $\mathbb{D}(\mathbf{Mod}A)$. Let $\mathcal{X}$ be an additive subcategory of $\mathbf{Mod}A$. Then we can form the homotopy category $\mathbb{K}(\mathcal{X})$, which is a triangulated subcategory (not closed under isomorphism) of $\mathbb{K}(A)$. This defines homotopy categories $\mathbb{K}(\operatorname{Proj} A), \mathbb{K}(\operatorname{Inj} A)$ and $\mathbb{K}(\operatorname{Flat} A)$ of projective, injective and flat A-modules, respectively.

If X is a ringed space and $\mathfrak{Mod}(X)$ the category of sheaves of modules on X, then $\mathbb{K}(X)$ denotes $\mathbb{K}(\mathfrak{Mod}X)$ and $\mathbb{D}(X)$ denotes $\mathbb{D}(\mathfrak{Mod}X)$. If X is a scheme then $\mathfrak{Qco}(X)$ denotes the category of quasi-coherent sheaves, $\mathbb{D}(\mathfrak{Qco}X)$ the derived category of quasi-coherent sheaves and $\mathbb{D}_{\mathrm{qc}}(X)$ the triangulated subcategory of $\mathbb{D}(X)$ consisting of the complexes with quasi-coherent cohomology.

Given an abelian category $\mathcal{A}$ an important operation on complexes in $\mathcal{A}$ is *truncation*. For a complex X in $\mathcal{A}$ and $n \in \mathbb{Z}$ our notation for the standard truncations are:

$$\begin{aligned}
X_{\leq n} &: \cdots \longrightarrow X^{n-2} \longrightarrow X^{n-1} \longrightarrow Ker\partial_X^n \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \\
X_{\geq n} &: \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow Coker\partial_X^{n-1} \longrightarrow X^{n+1} \longrightarrow X^{n+2} \longrightarrow \cdots \\
{}_bX_{\leq n} &: \cdots \longrightarrow X^{n-2} \longrightarrow X^{n-1} \longrightarrow X^n \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots \\
{}_bX_{\geq n} &: \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow X^n \longrightarrow X^{n+1} \longrightarrow X^{n+2} \longrightarrow \cdots
\end{aligned}$$

The complexes ${}_bX_{\leq n}$ and ${}_bX_{\geq n}$ are called the *brutal* or *stupid* truncations.

Remark 2.14. Let X, Y be complexes in an abelian category $\mathcal{A}$ and suppose that $Y^i = 0$ for $i \geq n$. Composition with the canonical morphism $X \longrightarrow {}_bX_{\leq n}$ defines an isomorphism

$$\operatorname{Hom}_{\mathbb{K}(\mathcal{A})}({}_bX_{\leq n}, Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}(\mathcal{A})}(X, Y)$$

On the other hand, if $Y^i = 0$ for $i \leq n$ then composition with the canonical morphism of complexes ${}_bX_{\geq n} \longrightarrow X$ gives an isomorphism $\operatorname{Hom}_{\mathbb{K}(\mathcal{A})}(X, Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}(\mathcal{A})}({}_bX_{\geq n}, Y)$. There are analogous results where we truncate in the second variable.

Given a morphism $f : X \longrightarrow Y$ of complexes we define the *mapping cone* complex $\operatorname{cone}(f)$ with the sign conventions of Conrad [Con00, §1.3]. For a morphism $f : X \longrightarrow Y$ in an arbitrary triangulated category $\mathcal{T}$ we refer to any object C completing f to a triangle $X \longrightarrow Y \longrightarrow C \longrightarrow \Sigma X$ in $\mathcal{T}$ as the *mapping cone* of f, by a standard abuse of notation.

Lemma 2.15. *Let $\mathcal{A}$ be an abelian category and suppose that we have a degree-wise split exact sequence of complexes in $\mathcal{A}$*

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0 \tag{2.6}$$

There exists a canonical morphism $z : Z \longrightarrow \Sigma X$ in $\mathbb{K}(\mathcal{A})$ fitting into a triangle in $\mathbb{K}(\mathcal{A})$

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{-z} \Sigma X \tag{2.7}$$

Proof. There is a factorization $Y \longrightarrow \mathrm{cone}(f) \longrightarrow Z$ of g through the mapping cone, and the degree-wise split exactness of (2.6) makes the factorization $\mu : \mathrm{cone}(f) \longrightarrow Z$ into a homotopy equivalence. Consider the canonical mapping cone triangle in $\mathbb{K}(\mathcal{A})$

$$X \xrightarrow{f} Y \xrightarrow{v} \mathrm{cone}(f) \xrightarrow{w} \Sigma X \tag{2.8}$$

and set $z = -w \circ \mu^{-1}$. The candidate triangle (2.7) is isomorphic to (2.8) and is therefore a triangle. The sign on z exists to ensure compatibility of $H^n(z)$ with the classical connecting morphism. $\square$

It is often useful to write a complex as the limit or colimit of bounded complexes, and in triangulated categories these limits and colimits become *homotopy* limits and colimits; see [Nee01b, §1.6] and [BN93] for relevant background. Since we are often working in homotopy categories, we will most often write complexes as homotopy (co)limits of their brutal truncations.

Remark 2.16. Let $\mathcal{A}$ be a cocomplete abelian category and suppose that we are given a sequence of degree-wise split monomorphisms of complexes in $\mathcal{A}$

$$X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow \cdots \tag{2.9}$$

We have a degree-wise split exact sequence, in the notation of [Nee01b, Definition 1.6.4]

$$0 \longrightarrow \oplus_{i \geq 1} X_i \xrightarrow{1-\mathrm{shift}} \oplus_{i \geq 1} X_i \longrightarrow \varinjlim X_i \longrightarrow 0$$

From Lemma 2.15 we deduce that the direct limit $\varinjlim X_i$ is isomorphic, in $\mathbb{K}(\mathcal{A})$, to the homotopy colimit of the sequence (2.9). There is an important special case: let W be a complex in $\mathcal{A}$ and for arbitrary $n \in \mathbb{Z}$ write W as the direct limit of the following sequence of brutal truncations

$$_bW_{\geq n} \longrightarrow {_bW_{\geq (n-1)}} \longrightarrow {_bW_{\geq (n-2)}} \longrightarrow \cdots \tag{2.10}$$

Each of these morphisms is a degree-wise split monomorphism, so W is isomorphic in $\mathbb{K}(\mathcal{A})$ to the homotopy colimit of the sequence (2.10). If $\mathcal{A}$ is complete rather than cocomplete, then W is the inverse limit of the following sequence of brutal truncations

$$\cdots \longrightarrow {_bW_{\leq (n+2)}} \longrightarrow {_bW_{\leq (n+1)}} \longrightarrow {_bW_{\leq n}} \tag{2.11}$$

and W is the homotopy limit in $\mathbb{K}(\mathcal{A})$ of this sequence.

Bicomplexes. Let $\mathcal{A}$ be a cocomplete abelian category. A *bicomplex* in $\mathcal{A}$ is a complex of complexes: it is a collection of objects $\{B^{ij}\}_{i,j \in \mathbb{Z}}$ and morphisms $\partial_1^{ij} : B^{ij} \longrightarrow B^{(i+1)j}, \partial_2^{ij} : B^{ij} \longrightarrow B^{i(j+1)}$ for $i, j \in \mathbb{Z}$ such that $\partial_1 \circ \partial_1 = 0, \partial_2 \circ \partial_2 = 0$ and $\partial_1 \circ \partial_2 = \partial_2 \circ \partial_1$. Represented on the page the first index is the column and the second the row; indices increase going to the right and upwards. The *totalization* of a bicomplex B is the complex $\mathrm{Tot}(B)$ defined by

$$\mathrm{Tot}(B)^n = \oplus_{i+j=n} B^{ij} \quad \text{with differential} \quad \partial^n u_{ij} = u_{(i+1)j} \partial_1^{ij} + (-1)^i u_{i(j+1)} \partial_2^{ij}$$

where u_{ij} is the injection of B^{ij} into the coproduct. A *morphism* of bicomplexes $\varphi : B \longrightarrow C$ is a collection of morphisms $\{\varphi^{ij} : B^{ij} \longrightarrow C^{ij}\}_{i,j\in\mathbb{Z}}$ with $\varphi\circ\partial_1 = \partial_1\circ\varphi$ and $\varphi\circ\partial_2 = \partial_2\circ\varphi$. Any such morphism induces a morphism of the totalizations $\mathrm{Tot}(\varphi) : \mathrm{Tot}(B) \longrightarrow \mathrm{Tot}(C)$ defined in degree n by $\mathrm{Tot}(\varphi)^n = \oplus_{i+j=n}\varphi^{ij}$, making the totalization into an additive functor from bicomplexes to complexes. The next lemma proves that this functor is exact.

Lemma 2.17. *Suppose that we have a short exact sequence of bicomplexes, split exact in each bidegree* $0 \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow 0$*. The sequence of totalizations*

$$0 \longrightarrow \mathrm{Tot}(B) \longrightarrow \mathrm{Tot}(C) \longrightarrow \mathrm{Tot}(D) \longrightarrow 0 \tag{2.12}$$

is split exact in each degree, and there is a canonical triangle in $\mathbb{K}(\mathcal{A})$

$$\mathrm{Tot}(B) \longrightarrow \mathrm{Tot}(C) \longrightarrow \mathrm{Tot}(D) \longrightarrow \Sigma\,\mathrm{Tot}(B)$$

Proof. Saying that the sequence is split exact in each bidegree means that for every $i, j \in \mathbb{Z}$ the sequence $0 \longrightarrow B^{ij} \longrightarrow C^{ij} \longrightarrow D^{ij} \longrightarrow 0$ is split exact. In this case it is not difficult to check that (2.12) is split exact in each degree, so from Lemma 2.15 we deduce the desired triangle. $\square$

Next we review some standard facts that tell us how to assemble the totalization of a bicomplex from its columns, or rows, via triangles in $\mathbb{K}(\mathcal{A})$. Let B be a bicomplex in $\mathcal{A}$. Given $k \in \mathbb{Z}$ we write $B^{\bullet k}$ for the kth row of the bicomplex and $B^{k\bullet}$ for the kth column. Let $B_{\mathrm{rows}\geq k}$ denote the bicomplex B with rows $< k$ deleted. Graphically, this is the following diagram

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & B^{(i-1)(k+1)} \arrow[r] \arrow[u] & B^{i(k+1)} \arrow[r] \arrow[u] & B^{(i+1)(k+1)} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & B^{(i-1)k} \arrow[r] \arrow[u] & B^{ik} \arrow[r] \arrow[u] & B^{(i+1)k} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & \cdots
\end{tikzcd} \tag{2.13}$$

There is a morphism of bicomplexes $B_{\mathrm{rows}\geq k+1} \longrightarrow B_{\mathrm{rows}\geq k}$ with cokernel $B^{\bullet k}$. If we agree that this row is placed in the correct vertical degree, then we have an exact sequence of bicomplexes $0 \longrightarrow B_{\mathrm{rows}\geq k+1} \longrightarrow B_{\mathrm{rows}\geq k} \longrightarrow B^{\bullet k} \longrightarrow 0$ which is split exact in each bidegree. This yields by Lemma 2.17 an exact sequence of the totalizations, split exact in each degree

$$0 \longrightarrow \mathrm{Tot}(B_{\mathrm{rows}\geq k+1}) \longrightarrow \mathrm{Tot}(B_{\mathrm{rows}\geq k}) \longrightarrow \Sigma^{-k}B^{\bullet k} \longrightarrow 0$$

from which we infer a canonical triangle in $\mathbb{K}(\mathcal{A})$

$$\mathrm{Tot}(B_{\mathrm{rows}\geq k+1}) \longrightarrow \mathrm{Tot}(B_{\mathrm{rows}\geq k}) \longrightarrow \Sigma^{-k}B^{\bullet k} \longrightarrow \Sigma\,\mathrm{Tot}(B_{\mathrm{rows}\geq k+1}) \tag{2.14}$$

and a sequence of morphisms of complexes, all degree-wise split monomorphisms

$$\mathrm{Tot}(B_{\mathrm{rows}\geq k}) \longrightarrow \mathrm{Tot}(B_{\mathrm{rows}\geq k-1}) \longrightarrow \mathrm{Tot}(B_{\mathrm{rows}\geq k-2}) \longrightarrow \cdots \tag{2.15}$$

The totalization of B is the direct limit of the totalizations $\mathrm{Tot}(B) = \varinjlim \mathrm{Tot}(B_{\mathrm{rows}\geq k-i})$. By Remark 2.16 this agrees with the homotopy colimit in $\mathbb{K}(\mathcal{A})$, so we have a triangle

$$\bigoplus_{i\geq 0} \mathrm{Tot}(B_{\mathrm{rows}\geq k-i}) \longrightarrow \bigoplus_{i\geq 0} \mathrm{Tot}(B_{\mathrm{rows}\geq k-i}) \longrightarrow \mathrm{Tot}(B) \longrightarrow \Sigma \bigoplus_{i\geq 0} \mathrm{Tot}(B_{\mathrm{rows}\geq k-i})$$

Remark 2.18. Suppose for example that B vanishes in rows $> k$. Then $B_{\mathrm{rows}\geq k} = B^{\bullet k}$ and we have constructed $\mathrm{Tot}(B)$ as the homotopy colimit of a sequence (2.15) beginning with the kth row and adding at each stage a new row, via the triangle of (2.14). In particular, if there is some localizing subcategory of $\mathbb{K}(\mathcal{A})$ (or $\mathbb{D}(\mathcal{A})$, provided $\mathcal{A}$ has exact coproducts) containing each row of B, then it must contain the totalization $\mathrm{Tot}(B)$.

We say that B is *bounded vertically* if there exist integers s, t with the row $B^{\bullet k}$ equal to zero unless $s \leq k \leq t$. In this case $\mathrm{Tot}(B_{\mathrm{rows}\geq i}) = \mathrm{Tot}(B)$ for $i \ll 0$, and a finite sequence of triangles (2.14) connects the row $B^{\bullet k}$ to the complex $\mathrm{Tot}(B)$. Hence, if any triangulated subcategory of $\mathbb{K}(\mathcal{A})$ or $\mathbb{D}(\mathcal{A})$ contains the rows of B, it contains the totalization $\mathrm{Tot}(B)$.

There is a similar technique for columns. Denote by $B_{\mathrm{cols}\geq k}$ the result of deleting all columns $< k$ in B. There is a morphism of bicomplexes $B_{\mathrm{cols}\geq k+1} \longrightarrow B_{\mathrm{cols}\geq k}$ with cokernel $B^{k\bullet}$ and an exact sequence of bicomplexes $0 \longrightarrow B_{\mathrm{cols}\geq k+1} \longrightarrow B_{\mathrm{cols}\geq k} \longrightarrow B^{k\bullet} \longrightarrow 0$ split exact in each bidegree, which yields a triangle in $\mathbb{K}(\mathcal{A})$

$$\mathrm{Tot}(B_{\mathrm{cols}\geq k+1}) \longrightarrow \mathrm{Tot}(B_{\mathrm{cols}\geq k}) \longrightarrow \Sigma^{-k} B^{k\bullet} \longrightarrow \Sigma\, \mathrm{Tot}(B_{\mathrm{cols}\geq k+1}) \tag{2.16}$$

Lemma 2.19. *Let $\mathcal{A}$ be a cocomplete abelian category and B a bicomplex in $\mathcal{A}$ that is bounded vertically and has contractible (acyclic) columns. Then* $\mathrm{Tot}(B)$ *is a contractible (acyclic) complex.*

Proof. Both claims are standard; we prove the statement about contractibility, since it is probably less well-known. Exactly the same argument shows that if the columns of B are acyclic then $\mathrm{Tot}(B)$ is acyclic. We observe that complex X in $\mathcal{A}$ is contractible if and only if it is acyclic and for every $n \in \mathbb{Z}$ the following short exact sequences is split exact

$$0 \longrightarrow Ker(\partial_X^n) \longrightarrow X^n \longrightarrow Ker(\partial_X^{n+1}) \longrightarrow 0$$

Since B is bounded vertically, the totalization complex $\mathrm{Tot}(B)$ in degree n only "sees" a horizontally bounded region of the bicomplex. To be precise, let s, t be integers such that the row $B^{\bullet k}$ is the zero complex unless $s \leq k \leq t$. Then

$$\mathrm{Tot}(B)^n = \oplus_{i+j=n} B^{ij} = B^{(n-s)s} \oplus \cdots \oplus B^{(n-t)t}$$

Thus in checking contractibility of $\mathrm{Tot}(B)$ we may as well assume that B is *horizontally* bounded as well as vertically bounded. In that case, there is an integer n such that the column $B^{k\bullet}$ is zero for $k > n$. Using (2.16) we have a series of triangles in $\mathbb{K}(\mathcal{A})$, beginning with $B_{\mathrm{cols}\geq n} = B^{n\bullet}$

$$\begin{gathered}\Sigma^{-n}B^{n\bullet} \longrightarrow \mathrm{Tot}(B_{\mathrm{cols}\geq n-1}) \longrightarrow \Sigma^{-n+1}B^{(n-1)\bullet} \longrightarrow \Sigma^{1-n}B^{n\bullet}\\ \mathrm{Tot}(B_{\mathrm{cols}\geq n-1}) \longrightarrow \mathrm{Tot}(B_{\mathrm{cols}\geq n-2}) \longrightarrow \Sigma^{-n+2}B^{(n-2)\bullet} \longrightarrow \Sigma\,\mathrm{Tot}(B_{\mathrm{cols}\geq n-1})\\ \vdots\end{gathered}$$

Since the columns are all contractible, we deduce that $\mathrm{Tot}(B_{\mathrm{cols}\geq n-i})$ is contractible for $i \geq 0$. But for $i \gg 0$ this totalization is equal to $\mathrm{Tot}(B)$, because B is horizontally bounded, which proves that $\mathrm{Tot}(B)$ is contractible. $\square$

2.2 Resolutions of Complexes

Given an abelian category $\mathcal{A}$ the correct notion of injective and projective resolutions for complexes was first elaborated by Spaltenstein [Spa88]. Following his notation, complexes in the orthogonal ${}^{\perp}\mathbb{K}_{\mathrm{ac}}(\mathcal{A}) \subseteq \mathbb{K}(\mathcal{A})$ are called $\mathbb{K}$-*projective*, and those in $\mathbb{K}_{\mathrm{ac}}(\mathcal{A})^{\perp} \subseteq \mathbb{K}(\mathcal{A})$ are called $\mathbb{K}$-*injective*. Using the isomorphism $H^n \operatorname{Hom}_{\mathcal{A}}(X, Y) \cong \operatorname{Hom}_{\mathbb{K}(\mathcal{A})}(X, \Sigma^n Y)$, we have the following alternative characterization:

$$\begin{aligned}\text{A complex } P \text{ is } \mathbb{K}\text{-projective } &\iff \operatorname{Hom}_{\mathcal{A}}(P, Z) \text{ is acyclic for every acyclic complex } Z\\ \text{A complex } I \text{ is } \mathbb{K}\text{-injective } &\iff \operatorname{Hom}_{\mathcal{A}}(Z, I) \text{ is acyclic for every acyclic complex } Z\end{aligned}$$

Any bounded above complex of projectives is $\mathbb{K}$-projective, and any bounded below complex of injectives is $\mathbb{K}$-injective. A $\mathbb{K}$-*projective resolution* of a complex X is a quasi-isomorphism $P \longrightarrow X$ from a $\mathbb{K}$-projective complex P, and a $\mathbb{K}$-*injective resolution* is a quasi-isomorphism $X \longrightarrow I$ to a $\mathbb{K}$-injective complex I.

If $\mathcal{A}$ is a category of modules over a ring, or sheaves of modules over a ringed space, a complex $\mathscr{X}$ in $\mathcal{A}$ is called $\mathbb{K}$-*flat* if $\mathscr{X} \otimes \mathscr{E}$ is acyclic for any acyclic complex $\mathscr{E}$ in $\mathcal{A}$. The $\mathbb{K}$-flat complexes form a localizing subcategory of $\mathbb{K}(\mathcal{A})$, any bounded above complex of flats is $\mathbb{K}$-flat and the class of $\mathbb{K}$-flat complexes is closed under direct limits and homotopy colimits in $\mathbb{K}(\mathcal{A})$; see [Spa88, §5] and [Lip, §2.5]. A $\mathbb{K}$-*flat resolution* of $\mathscr{X}$ is a quasi-isomorphism $\mathscr{F} \longrightarrow \mathscr{X}$ from a $\mathbb{K}$-flat complex $\mathscr{F}$. Let us clear up a possible point of confusion: given a scheme X and a complex $\mathscr{X}$ of quasi-coherent sheaves, we say that $\mathscr{X}$ is $\mathbb{K}$-flat when it is $\mathbb{K}$-flat as a complex of sheaves of modules in the sense just defined.

Be careful to observe that the definitions of $\mathbb{K}$-projectivity and $\mathbb{K}$-injectivity are relative to the abelian category $\mathcal{A}$. If we have an abelian subcategory $\mathcal{B} \subseteq \mathcal{A}$ the $\mathbb{K}$-injectives in the two categories may differ, and this is a distinction to keep in mind when we come to study categories of quasi-coherent sheaves $\mathfrak{Qco}(X) \subseteq \mathfrak{Mod}(X)$ for a scheme X.

Remark 2.20. A complex X has a $\mathbb{K}$-injective resolution if it fits into a triangle in $\mathbb{K}(\mathcal{A})$

$$C \longrightarrow X \longrightarrow I \longrightarrow \Sigma C \tag{2.17}$$

with $C \in \mathbb{K}_{\mathrm{ac}}(\mathcal{A})$ and $I \in \mathbb{K}_{\mathrm{ac}}(\mathcal{A})^{\perp}$. If this triangle exists for every complex X we say that $\mathcal{A}$ *has* $\mathbb{K}$*-injective resolutions*. By Lemma 2.3 it is equivalent to say that the Verdier quotient $q : \mathbb{K}(\mathcal{A}) \longrightarrow \mathbb{D}(\mathcal{A})$ has a right adjoint, or that there is a localization sequence

$$\begin{tikzcd} \mathbb{K}_{\mathrm{ac}}(\mathcal{A}) \arrow[r, shift left] & \mathbb{K}(\mathcal{A}) \arrow[l, shift left] \arrow[r, shift left, "q"] & \mathbb{D}(\mathcal{A}) \arrow[l, shift left, "q_\rho"] \end{tikzcd}$$

The right adjoint q_ρ sends $X \in \mathbb{D}(\mathcal{A})$ to the complex I fitting into a triangle (2.17), which is unique up to homotopy equivalence. To be perfectly clear, the adjoint q_ρ *takes* $\mathbb{K}$*-injective resolutions*. It is well-known that $\mathbb{K}$-injective resolutions exist for any Grothendieck abelian category $\mathcal{A}$; see [Spa88] and [AJS00]. There is a dual discussion of $\mathbb{K}$-projective resolutions, when they exist.

There is a standard construction of resolutions of unbounded complexes that we review below. In what follows $\mathcal{A}$ denotes a Grothendieck abelian category; see [Ste75, Chapter 5] for the definition. Let $\mathcal{P} \subseteq \mathcal{A}$ be a class of objects that is closed under isomorphism and arbitrary coproducts, contains the zero objects, and has the property that every $X \in \mathcal{A}$ admits an epimorphism $P \longrightarrow X$ with $P \in \mathcal{P}$. We say that a *complex* P is *in* $\mathcal{P}$ when $P^i \in \mathcal{P}$ for every $i \in \mathbb{Z}$. The dual of [Har66, Lemma 4.6] constructs, for any bounded above complex X in $\mathcal{A}$, a quasi-isomorphism $P \longrightarrow X$ with P a bounded above complex in $\mathcal{P}$. The next lemma proves that, if we choose our resolutions correctly, we can make this process functorial. This seems to be due to Spaltenstein; see [Spa88, Lemma 3.3].

Lemma 2.21. *Let $X \longrightarrow Y$ be a morphism of bounded above complexes in $\mathcal{A}$, and suppose $P \longrightarrow X$ is a quasi-isomorphism with P a bounded above complex in $\mathcal{P}$. There exists a commutative diagram in $\mathbb{K}(\mathcal{A})$*

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Q \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \tag{2.18}$$

with $Q \longrightarrow Y$ a quasi-isomorphism and Q a bounded above complex in $\mathcal{P}$.

Proof. Let T be the mapping cone of $P \longrightarrow X \longrightarrow Y$. There is a canonical morphism $\Sigma^{-1}T \longrightarrow P$ and we can find a quasi-isomorphism $P' \longrightarrow \Sigma^{-1}T$ with P' a bounded above complex in $\mathcal{P}$. Take Q to be the mapping cone of $P' \longrightarrow \Sigma^{-1}T \longrightarrow P$. □

Using homotopy colimits, we can construct a resolution for any complex.

Lemma 2.22. *Any complex X in $\mathcal{A}$ admits a quasi-isomorphism $P \longrightarrow X$ with P a complex in $\mathcal{P}$ that is the homotopy colimit in $\mathbb{K}(\mathcal{A})$ of a sequence*

$$P_0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow P_3 \longrightarrow \cdots$$

of bounded above complexes in $\mathcal{P}$.

Proof. By the dual of [Har66, Lemma 4.6] we can find a quasi-isomorphism $P_0 \longrightarrow X_{\leq 0}$ with P_0 a bounded above complex in $\mathcal{P}$ and, using Lemma 2.21, construct a commutative diagram in $\mathbb{K}(\mathcal{A})$

$$\begin{tikzcd} P_0 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_2 \arrow[r] \arrow[d] & \cdots \\ X_{\leq 0} \arrow[r] & X_{\leq 1} \arrow[r] & X_{\leq 2} \arrow[r] & \cdots \end{tikzcd} \tag{2.19}$$

in which every vertical morphism is a quasi-isomorphism, and each P_i is a bounded above complex in $\mathcal{P}$. Taking the homotopy colimit of the rows of the diagram (2.19) in $\mathbb{K}(\mathcal{A})$ we obtain a quasi-isomorphism $\underrightarrow{holim}_{n\geq 0} P_n \longrightarrow X$ (using exactness of coproducts in $\mathcal{A}$). Since $P = \underrightarrow{holim}_{n\geq 0} P_n$ is, by definition, the mapping cone of a morphism of two complexes in $\mathcal{P}$, it is also a complex in $\mathcal{P}$. □

2.3 Relative Homological Algebra

Throughout this section A is a ring (commutative, as usual) and modules are defined over A by default. For a variety of reasons, we are interested in the homotopy categories of injective, projective and flat modules

$$\mathbb{K}(\operatorname{Inj} A), \quad \mathbb{K}(\operatorname{Proj} A), \quad \mathbb{K}(\operatorname{Flat} A)$$

so it is worthwhile studying ways to construct compact objects in these categories. The subject of relative homological algebra provides a rich set of tools for this purpose: there is a unified way of constructing, from a finitely generated module M, a compact object in $\mathbb{K}(\mathscr{X})$ for $\mathscr{X}$ one of $\operatorname{Inj}(A), \operatorname{Proj}(A), \operatorname{Flat}(A)$.

In this section we introduce some relevant concepts from the literature, and explain how to view results of relative homological algebra in the context of homotopy categories. Many of these ideas have now appeared in a comprehensive paper of Holm and Jørgensen [HJ07] which treats the material in greater generality than we need to here. The observations of this section were obtained independently of their paper.

Setup. In this section $\mathscr{X}$ denotes a class of modules closed under isomorphism, finite direct sums and direct summands. In practice, $\mathscr{X}$ will be one of the classes $\operatorname{Inj}(A), \operatorname{Proj}(A)$ or $\operatorname{Flat}(A)$. We denote by $\mathbb{K}(\mathscr{X})$ the corresponding homotopy category.

In constructing objects of $\mathbb{K}(\mathscr{X})$ the key concept is that of a *preenvelope*, which has been studied extensively in the literature; we recall the basic definitions from [Xu96, §1.2].

Definition 2.23. An $\mathscr{X}$*-preenvelope* of a module M is a morphism $\phi : M \longrightarrow X$ with $X \in \mathscr{X}$ such that any morphism $f : M \longrightarrow X'$ where $X' \in \mathscr{X}$ factors as $f = g\phi$ for some morphism $g : X \longrightarrow X'$. We do *not* require the factorization to be unique. Taking $\mathscr{X}$ to be the classes $\operatorname{Inj}(A), \operatorname{Proj}(A)$ and $\operatorname{Flat}(A)$ we obtain, respectively, the notion of a *injective, projective* and *flat preenvelope*. Some observations:

- An injective preenvelope is precisely a monomorphism $\phi : M \longrightarrow X$ with X injective, so every module M has an injective preenvelope.

- Over a noetherian ring every module has a flat preenvelope [Xu96, Theorem 2.5.1] and every finitely generated module has a projective preenvelope (see Remark 2.26).

Even over a noetherian ring not every module has a projective preenvelope, and those projective or flat preenvelopes that exist are not always monomorphisms; for example, see [AM93, Corollary 3.6] and [Din96, Corollary 3.9].

A flat resolution of a module M is an exact sequence with each F^i flat

$$\cdots \longrightarrow F^{-1} \longrightarrow F^0 \longrightarrow M \longrightarrow 0 \tag{2.20}$$

Over a noetherian ring we can take flat preenvelopes and cokernels repeatedly to construct a complex (not necessarily exact) extending to the right with each F^i flat

$$0 \longrightarrow M \longrightarrow F^0 \longrightarrow F^1 \longrightarrow F^2 \longrightarrow \cdots \tag{2.21}$$

This is called a *proper right flat resolution* of M (see below for the precise definition). Such resolutions were introduced by Enochs in [Eno81] where they were known as *resolvents*. In this article, we use the notation of Holm [Hol04, §2.1]; see also [Xu96, §3.6] and [EJ00, §8.1]. We prove that applying the construction (2.21) to finitely generated modules produces compact objects in $\mathbb{K}(\operatorname{Flat} A)$ (resp. $\mathbb{K}(\operatorname{Proj} A)$, when we use projective preenvelopes).

Definition 2.24. An *augmented proper right $\mathcal{X}$-resolution* of a module M is a complex

$$S : 0 \longrightarrow M \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots \tag{2.22}$$

with $X^i \in \mathcal{X}$ such that $\operatorname{Hom}_A(S, X)$ is acyclic for every $X \in \mathcal{X}$. We call the complex X consisting of just the objects X^i a *proper right $\mathcal{X}$-resolution* of M. Note that the complex S in (2.22) need not be exact.

A proper right injective resolution is simply an injective resolution. Over a noetherian ring every module has a proper right flat resolution because flat preenvelopes exist; see the next remark. We show in Remark 2.26 that over a noetherian ring a finitely generated module has a proper right projective resolution.

Remark 2.25. Let M be a module with an augmented proper right $\mathcal{X}$-resolution S

$$S : 0 \longrightarrow M \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots \tag{2.23}$$

Then the morphisms $M \longrightarrow X^0$ and $Coker(M \longrightarrow X^0) \longrightarrow X^1, Coker(X^{i-1} \longrightarrow X^i) \longrightarrow X^{i+1}$ for $i \geq 1$ are $\mathcal{X}$-preenvelopes. In fact this property characterizes the complexes S of the form given in (2.23) that are augmented proper right $\mathcal{X}$-resolutions.

On the other hand, suppose that every module has an $\mathcal{X}$-preenvelope and let M be a module. We construct an augmented proper right $\mathcal{X}$-resolution of M as follows: take an $\mathcal{X}$-preenvelope $M \longrightarrow X^0$ with cokernel $X^0 \longrightarrow C^0$, then take an $\mathcal{X}$-preenvelope $C^0 \longrightarrow X^1$ and let C^1 denote the cokernel of the composite $X^0 \longrightarrow C^0 \longrightarrow X^1$. Take an $\mathcal{X}$-preenvelope of C^1 and repeat to construct an augmented proper right $\mathcal{X}$-resolution.

Remark 2.26. Let A be a noetherian ring and M a finitely generated module. Then the dual $M^* = \mathrm{Hom}_A(M, A)$ is a finitely generated module, so it admits a resolution by finitely generated projectives (in the ordinary sense)

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M^* \longrightarrow 0$$

Applying $\mathrm{Hom}_A(-, A)$ and composing with the canonical morphism $M \longrightarrow M^{**}$ we have a complex of finitely generated modules with each P_i^* a finitely generated projective module

$$S : 0 \longrightarrow M \longrightarrow P_0^* \longrightarrow P_1^* \longrightarrow P_2^* \longrightarrow \cdots \tag{2.24}$$

It follows from [Jør05, Lemma 1.3] that S is an augmented proper right projective resolution; that is, P^* is a proper right projective resolution of M. In particular, the morphism $M \longrightarrow P_0^*$ is a projective preenvelope.

The next result gives the connection between proper right resolutions and orthogonals.

Proposition 2.27. *Let M be a module and suppose that we are given a complex*

$$S : 0 \longrightarrow M \longrightarrow X^0 \longrightarrow X^1 \longrightarrow X^2 \longrightarrow \cdots$$

with $X^i \in \mathcal{X}$. The following conditions are equivalent:

(i) S is an augmented proper right $\mathcal{X}$-resolution.

(ii) S belongs to ${}^{\perp}\mathbb{K}(\mathcal{X})$. That is, $\mathrm{Hom}_{\mathbb{K}(A)}(S, Z) = 0$ for every $Z \in \mathbb{K}(\mathcal{X})$.

Proof. $(i) \Rightarrow (ii)$ By definition, $\mathrm{Hom}_A(S, X)$ is acyclic for $X \in \mathcal{X}$. Taking cohomology we deduce that $\mathrm{Hom}_{\mathbb{K}(A)}(\Sigma^i S, X) = 0$ for every $i \in \mathbb{Z}$, so X belongs to $\{S\}^{\perp}$. It follows by a standard argument that any bounded complex in $\mathbb{K}(\mathcal{X})$ also belongs to $\{S\}^{\perp}$.

By Remark 2.16 every bounded below complex Z in $\mathbb{K}(\mathcal{X})$ is the homotopy limit of its truncations ${}_bZ_{\leq n}$, which are bounded complexes and therefore belong to $\{S\}^{\perp}$. This orthogonal is colocalizing, so it is closed under homotopy limits, and thus Z is in $\{S\}^{\perp}$. Finally, given any complex $Z \in \mathbb{K}(\mathcal{X})$, we have by Remark 2.14

$$\mathrm{Hom}_{\mathbb{K}(A)}(S, Z) \cong \mathrm{Hom}_{\mathbb{K}(A)}(S, {}_bZ_{\geq -1}) = 0$$

which shows that S belongs to ${}^{\perp}\mathbb{K}(\mathcal{X})$, as required.

$(ii) \Rightarrow (i)$ For $X \in \mathcal{X}$ we have $H^i \mathrm{Hom}_A(S, X) \cong \mathrm{Hom}_{\mathbb{K}(A)}(S, \Sigma^i X)$ which is zero by assumption. This proves that $\mathrm{Hom}_A(S, X)$ is acyclic and that S is an augmented proper right $\mathcal{X}$-resolution. □

A proper right $\mathcal{X}$-resolution of M is the closest approximation to M among complexes in $\mathbb{K}(\mathcal{X})$. More precisely, such a resolution *represents* M in the homotopy category $\mathbb{K}(\mathcal{X})$. It follows that the proper right $\mathcal{X}$-resolution X_M of M (if it exists) is unique up to canonical isomorphism in $\mathbb{K}(\mathcal{X})$.

Corollary 2.28. *Let M be a module with proper right $\mathcal{X}$-resolution X_M. The canonical morphism of complexes $M \longrightarrow X_M$ extends to a triangle in $\mathbb{K}(A)$*

$$S \longrightarrow M \longrightarrow X_M \longrightarrow \Sigma S \tag{2.25}$$

with S in ${}^{\perp}\mathbb{K}(\mathcal{X})$. For any complex Y in $\mathbb{K}(\mathcal{X})$ there is a natural isomorphism

$$\operatorname{Hom}_{\mathbb{K}(\mathcal{X})}(X_M, Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}(A)}(M, Y) \tag{2.26}$$

Proof. Let S be the augmented proper right $\mathcal{X}$-resolution of M corresponding to X_M, where we agree that in the complex S the module M has degree zero. Then ΣS is canonically isomorphic to the mapping cone of $M \longrightarrow X_M$ and fits into a triangle (2.25). By Proposition 2.27 we have $S \in {}^{\perp}\mathbb{K}(\mathcal{X})$, so given $Y \in \mathbb{K}(\mathcal{X})$ we can apply the homological functor $\operatorname{Hom}_{\mathbb{K}(A)}(-, Y)$ to (2.25) to deduce the isomorphism (2.26). □

Let us state this result for our special examples of the class $\mathcal{X}$. Let A be a noetherian ring and M a module with injective resolution I_M and proper right flat resolution F_M. By the previous corollary we have isomorphisms

$$\operatorname{Hom}_{\mathbb{K}(\mathrm{Inj}\,A)}(I_M, -) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}(A)}(M, -) \tag{2.27}$$

$$\operatorname{Hom}_{\mathbb{K}(\mathrm{Flat}\,A)}(F_M, -) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}(A)}(M, -) \tag{2.28}$$

If M is finitely generated then it has a proper right projective resolution P_M and

$$\operatorname{Hom}_{\mathbb{K}(\mathrm{Proj}\,A)}(P_M, -) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}(A)}(M, -) \tag{2.29}$$

A finitely generated module M is compact in $\mathbb{K}(A)$, so in this case the proper resolutions I_M, P_M and F_M are compact in their respective homotopy categories (here we use the fact that these choices of $\mathcal{X}$ are closed under coproducts). Letting M vary produces a compact generating set for $\mathbb{K}(\mathrm{Inj}\,A)$ and $\mathbb{K}(\mathrm{Proj}\,A)$; see [Kra05, Proposition 2.3] and Theorem 2.30.

These isomorphisms are not new; in the injective case we have reproduced [Kra05, Lemma 2.1] and in the projective case [Jør05, Lemma 1.5], since the proper right projective resolution P_M is precisely the complex P^* of Jørgensen's [Jør05, Construction 1.2]. The point of this section is that these results are special cases of a general principle:

Meta-Theorem 2.29. *To represent a module M in the homotopy category $\mathbb{K}(\mathcal{X})$ take a proper right $\mathcal{X}$-resolution of M. To construct compact objects in $\mathbb{K}(\mathcal{X})$, represent finitely generated modules.*

The next result is an improvement on a result of Jørgensen [Jør05, Theorem 2.4] made possible by recent work of Neeman[1]. To be clear, nothing about the next theorem is new; the fact that $\mathbb{K}(\mathrm{Proj}\,A)$ is compactly generated can be found as [Nee06a, Proposition 6.14]. We state the result here for the reader's convenience.

[1]We should clarify what we mean by an "improvement". Jørgensen's theorem is, in some sense, more general, but specialized to commutative noetherian rings he requires, roughly speaking, that the ring have finite Krull dimension. See [Jør05] for the precise details.

Theorem 2.30. *If A is noetherian then $\mathbb{K}(\operatorname{Proj} A)$ is compactly generated, and*

$$\mathcal{S} = \{\Sigma^i P_M \mid M \textit{ is a finitely generated module and } i \in \mathbb{Z}\}$$

is a compact generating set, where P_M denotes a proper right projective resolution of M.

Proof. Up to isomorphism there is a set of finitely generated modules. Pick one module from each isomorphism class and let $\mathcal{S}$ be a set containing a proper right projective resolution for each of these chosen modules, together with all shifts of such complexes. These objects are compact, and we claim that $\mathcal{S}$ generates $\mathbb{K}(\operatorname{Proj} A)$. To prove the claim, we have to show that if a complex Q of projective modules satisfies

$$\operatorname{Hom}_{\mathbb{K}(\operatorname{Proj} A)}(\Sigma^i P_M, Q) = 0 \quad \text{for all finitely generated } M \text{ and } i \in \mathbb{Z} \tag{2.30}$$

then Q is zero in $\mathbb{K}(\operatorname{Proj} A)$. The proof of [Jør05, Theorem 2.4] shows that the condition (2.30) forces Q to be an acyclic complex with flat kernels. At this point Jørgensen uses his hypothesis that all flat modules have finite projective dimension to conclude that each kernel module of Q is projective, which implies that Q is contractible.

This hypothesis is not necessary: Neeman proves in [Nee06a, Theorem 7.7] that any acyclic complex of flat modules with flat kernels belongs to the orthogonal $\mathbb{K}(\operatorname{Proj} A)^\perp$ as an object of $\mathbb{K}(\operatorname{Flat} A)$. Since Q belongs to both $\mathbb{K}(\operatorname{Proj} A)$ and $\mathbb{K}(\operatorname{Proj} A)^\perp$ it must be zero, which is what we needed to show. □

Remark 2.31. Let A be a noetherian ring. Given a finitely generated module M, we can construct a proper right flat resolution F_M of M with each F_M^i finitely generated and projective [EJ85, Example 3.4]. By (2.28) we have a natural isomorphism

$$\operatorname{Hom}_{\mathbb{K}(\operatorname{Proj} A)}(F_M, -) = \operatorname{Hom}_{\mathbb{K}(\operatorname{Flat} A)}(F_M, -) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}(A)}(M, -)$$

This is the unique property of the proper right projective resolution, so we deduce that F_M is isomorphic in $\mathbb{K}(A)$ to the proper right projective resolution of M.

This is consistent with a result of Neeman [Nee06a, Remark 6.13] which says that the inclusion $\mathbb{K}(\operatorname{Proj} A) \longrightarrow \mathbb{K}(\operatorname{Flat} A)$ preserves compactness: by Theorem 2.30 there is a compact generating set for $\mathbb{K}(\operatorname{Proj} A)$ consisting of proper right projective resolutions P_M of finitely generated modules M and, as we have just observed, such a complex is compact in $\mathbb{K}(\operatorname{Flat} A)$ as it agrees with the proper right flat resolution.

There is a dual theory of *precovers* and *proper left resolutions* that can be used to contravariantly represent a module in $\mathbb{K}(\mathcal{X})$. Our reference is once again [Xu96, §1.2].

Definition 2.32. An *$\mathcal{X}$-precover* of a module M is a morphism $\phi : X \longrightarrow M$ with $X \in \mathcal{X}$ such that any morphism $f : X' \longrightarrow M$ with $X' \in \mathcal{X}$ factors as $f = \phi g$ for some $g : X' \longrightarrow X$. We do *not* require the factorization to be unique. Taking $\mathcal{X}$ to be the classes $\operatorname{Inj}(A), \operatorname{Proj}(A)$ and $\operatorname{Flat}(A)$ we obtain, respectively, the notion of a *injective, projective* and *flat precover*. Some observations:

- A projective precover is precisely an epimorphism $\phi : X \longrightarrow M$ with X projective, so every module M has a projective precover.

- Over a noetherian ring every module has an injective precover; see [Xu96, Theorem 2.4.1].

- Every module has a flat precover. This was a conjecture of Enochs for almost twenty years and was settled in the affirmative by Bican, El Bashir and Enochs [BEB01].

The reader should note that flat precovers are always epimorphisms, but injective precovers are not necessarily epimorphisms and can even be zero; see [Xu96, Theorem 2.4.8].

Definition 2.33. An *augmented proper left $\mathcal{X}$-resolution* of a module M is a complex

$$S : \cdots \longrightarrow X^{-2} \longrightarrow X^{-1} \longrightarrow X^0 \longrightarrow M \longrightarrow 0 \tag{2.31}$$

with $X^i \in \mathcal{X}$ such that $\mathrm{Hom}_A(X, S)$ is acyclic for every $X \in \mathcal{X}$. We call the complex X consisting of just the objects X^i a *proper left $\mathcal{X}$-resolution* of M. Following tradition, we will sometimes index the complex X homologically rather than cohomologically (that is, writing X_i for X^{-i}). Note that the complex S in (2.31) need not be exact.

A proper left projective resolution is simply a projective resolution. Every module has a proper left flat resolution and over a noetherian ring every module has a proper left injective resolution, because precovers of both type exist; see the next remark.

Remark 2.34. Let M be a module with augmented proper left $\mathcal{X}$-resolution

$$S : \cdots \longrightarrow X^{-2} \longrightarrow X^{-1} \longrightarrow X^0 \longrightarrow M \longrightarrow 0 \tag{2.32}$$

Then the morphisms $X^0 \longrightarrow M$ and $X^{-1} \longrightarrow Ker(X^0 \longrightarrow M)$ and $X^{-i-1} \longrightarrow Ker(\partial_X^{-i})$ for $i \geq 1$ are $\mathcal{X}$-precovers. In fact this property characterizes the complexes S of the form given in (2.32) that are augmented proper left $\mathcal{X}$-resolutions. If every module has an $\mathcal{X}$-precover then we can take repeated precovers and kernels to construct an augmented proper left $\mathcal{X}$-resolution of M.

The next pair of results give the connection between left resolutions and orthogonals. The results are dual to Proposition 2.27 and Corollary 2.28, so we omit the proofs.

Proposition 2.35. *Let M be a module and suppose that we are given a complex*

$$S : \cdots \longrightarrow X^{-2} \longrightarrow X^{-1} \longrightarrow X^0 \longrightarrow M \longrightarrow 0$$

with $X^i \in \mathcal{X}$. The following conditions are equivalent:

(i) S is an augmented proper left $\mathcal{X}$-resolution.

(ii) S belongs to $\mathbb{K}(\mathcal{X})^{\perp}$. That is, $\mathrm{Hom}_{\mathbb{K}(A)}(Z, S) = 0$ *for every $Z \in \mathbb{K}(\mathcal{X})$.*

A proper left $\mathcal{X}$-resolution *contravariantly represents* a module in $\mathbb{K}(\mathcal{X})$.

Corollary 2.36. *Let M be a module with proper left $\mathcal{X}$-resolution X_M. Then the canonical morphism of complexes $X_M \longrightarrow M$ extends to a triangle in $\mathbb{K}(A)$*

$$X_M \longrightarrow M \longrightarrow S \longrightarrow \Sigma X_M \tag{2.33}$$

with S in $\mathbb{K}(\mathcal{X})^{\perp}$. For any complex Y in $\mathbb{K}(\mathcal{X})$ there is a natural isomorphism

$$\operatorname{Hom}_{\mathbb{K}(\mathcal{X})}(Y, X_M) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}(A)}(Y, M) \tag{2.34}$$

The triangle (2.25) of Corollary 2.28 can be taken as the *definition* of a proper right resolution. This has the advantage that the generalization to complexes is immediate; we will make use of proper resolutions of complexes in Chapter 4 and again in Appendix B.

Definition 2.37. Given a complex M of modules and a triangle in $\mathbb{K}(A)$

$$S \longrightarrow M \longrightarrow X_M \longrightarrow \Sigma S \tag{2.35}$$

with $X_M \in \mathbb{K}(\mathcal{X})$ and $S \in {}^{\perp}\mathbb{K}(\mathcal{X})$ we call the complex X_M a *proper right $\mathcal{X}$-resolution*. The complexes M fitting into a triangle of the form (2.35) form a triangulated subcategory of $\mathbb{K}(A)$ (called the Verdier sum). Hence, if every module has a $\mathcal{X}$-preenvelope then every bounded complex of modules admits a proper right $\mathcal{X}$-resolution. Similarly, *a proper left $\mathcal{X}$-resolution* of a complex M is a morphism of complexes $X_M \longrightarrow M$ fitting into a triangle

$$X_M \longrightarrow M \longrightarrow S \longrightarrow \Sigma X_M$$

with $X_M \in \mathbb{K}(\mathcal{X})$ and $S \in \mathbb{K}(\mathcal{X})^{\perp}$. When M is a module these two definitions agree with the original definitions; see Corollary 2.28 and Corollary 2.36.

Chapter 3

The Mock Homotopy Category of Projectives

In this chapter we introduce the mock homotopy category $\mathbb{K}_m(\operatorname{Proj} X)$ of projectives and give its basic properties. In Section 3.1 we define the Čech triangles, enabling us to pass from local statements to global ones; an important example is Theorem 3.16, where we show that $\mathbb{K}_m(\operatorname{Proj} X)$ has small Homs. In Section 3.2 we prove that every quasi-coherent sheaf is a quotient of a flat quasi-coherent sheaf, a fact that will be needed in the sequel.

Setup. In this chapter X denotes a scheme, and sheaves are defined over X by default.

Definition 3.1. Let $\mathbb{K}(\operatorname{Flat} X)$ be the homotopy category of flat quasi-coherent sheaves. Its objects are the complexes of flat quasi-coherent sheaves

$$\cdots \longrightarrow \mathscr{F}^{n-1} \xrightarrow{\partial^{n-1}} \mathscr{F}^{n} \xrightarrow{\partial^{n}} \mathscr{F}^{n+1} \longrightarrow \cdots$$

and its morphisms are the homotopy equivalence classes of morphisms of complexes. The category $\mathbb{K}(\operatorname{Flat} X)$ is a triangulated category with coproducts.

Let $\mathbb{E}(X)$ denote the full subcategory of $\mathbb{K}(\operatorname{Flat} X)$ consisting of complexes $\mathscr{E}$ with the property that $\mathscr{F} \otimes \mathscr{E}$ is acyclic for every sheaf of modules $\mathscr{F}$. Taking $\mathscr{F} = \mathscr{O}_X$ shows that all such complexes are acyclic. It is worth mentioning the following fact, though we will make no use of it: in the definition of $\mathbb{E}(X)$ it is equivalent to require that $\mathscr{F} \otimes \mathscr{E}$ be acyclic for every *quasi-coherent* sheaf $\mathscr{F}$; see Lemma 3.25.

Lemma 3.2. $\mathbb{E}(X)$ *is a localizing subcategory of* $\mathbb{K}(\operatorname{Flat} X)$.

Proof. Given a triangle $\mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{C} \longrightarrow \Sigma\mathscr{A}$ in $\mathbb{K}(\operatorname{Flat} X)$ with $\mathscr{A}$ and $\mathscr{B}$ in $\mathbb{E}(X)$ we have to show that $\mathscr{C} \in \mathbb{E}(X)$. For any sheaf of modules $\mathscr{F}$ there is a triangle in the homotopy category $\mathbb{K}(X)$ of sheaves of modules

$$\mathscr{F} \otimes \mathscr{A} \longrightarrow \mathscr{F} \otimes \mathscr{B} \longrightarrow \mathscr{F} \otimes \mathscr{C} \longrightarrow \Sigma(\mathscr{F} \otimes \mathscr{A})$$

From the long exact cohomology sequence we infer that $\mathscr{F} \otimes \mathscr{C}$ is acyclic, thus $\mathscr{C} \in \mathbb{E}(X)$. Tensoring with a complex preserves coproducts, so $\mathbb{E}(X)$ is a localizing subcategory. $\square$

Definition 3.3. Let $\mathbb{K}_m(\operatorname{Proj} X)$ be the Verdier quotient

$$\mathbb{K}_m(\operatorname{Proj} X) = \mathbb{K}(\operatorname{Flat} X)/\mathbb{E}(X)$$

which is a triangulated category with coproducts; the objects are the complexes of flat quasi-coherent sheaves and a morphism $\mathscr{F} \longrightarrow \mathscr{G}$ in $\mathbb{K}_m(\operatorname{Proj} X)$ is an equivalence class of "fractions" given by pairs of morphisms in $\mathbb{K}(\operatorname{Flat} X)$

$$\mathscr{F} \xleftarrow{\ f\ } \mathscr{W} \xrightarrow{\ g\ } \mathscr{G}$$

where f has mapping cone in $\mathbb{E}(X)$, and is in particular a quasi-isomorphism. The quotient functor $Q : \mathbb{K}(\operatorname{Flat} X) \longrightarrow \mathbb{K}_m(\operatorname{Proj} X)$ preserves coproducts [Nee01b, Corollary 3.2.11] and sends morphisms with mapping cone in $\mathbb{E}(X)$ to isomorphisms in $\mathbb{K}_m(\operatorname{Proj} X)$. It is universal with this property.

The subscript "m" for *mock* is a reminder that $\mathbb{K}_m(\operatorname{Proj} X)$ is *not* defined to be the homotopy category of any additive category. In particular, it is not the homotopy category of projective quasi-coherent sheaves. There is no danger of confusion, because we do not consider such sheaves.

The next result gives several characterizations of the complexes in $\mathbb{E}(X)$. In the proof we use the Tor sheaves, so it is worth reminding the reader of the definition. For complexes $\mathscr{F}, \mathscr{G}$ of sheaves of modules there is a tensor complex $\mathscr{F} \otimes \mathscr{G}$ defined in degree $n \in \mathbb{Z}$ by $(\mathscr{F} \otimes \mathscr{G})^n = \bigoplus_{i+j=n} \mathscr{F}^i \otimes \mathscr{G}^j$ with an appropriate differential; consult [Lip, (1.5.4)]. The derived tensor product is a bifunctor on the derived category $\mathbb{D}(X)$ of (arbitrary) sheaves of modules, which is triangulated in each variable

$$- \underset{=}{\otimes} - : \mathbb{D}(X) \times \mathbb{D}(X) \longrightarrow \mathbb{D}(X)$$

and defined by $\mathscr{F} \underset{=}{\otimes} \mathscr{G} = \mathscr{F} \otimes F_{\mathscr{G}}$ where $F_{\mathscr{G}}$ is a $\mathbb{K}$-flat resolution of $\mathscr{G}$; see [Lip, §2.5]. In the next result, and throughout this thesis, we will speak often of $\mathbb{K}$-flat complexes, which are the generalization of flat modules; see Section 2.2 for the definition.

If $\mathscr{F}, \mathscr{G}$ are *sheaves* of modules, we have for $i \in \mathbb{Z}$ a sheaf $\mathscr{T}\!or_i(\mathscr{F}, \mathscr{G}) = H^{-i}(\mathscr{F} \underset{=}{\otimes} \mathscr{G})$. The sheaf $\mathscr{F}$ is flat if and only if $\mathscr{T}\!or_i(\mathscr{F}, \mathscr{G}) = 0$ for every sheaf of modules $\mathscr{G}$ and $i > 0$.

Proposition 3.4. *Let $\mathscr{E}$ be a complex of flat quasi-coherent sheaves. The following are equivalent:*

(i) $\mathscr{E}$ belongs to $\mathbb{E}(X)$.

(ii) $\mathscr{E}$ is acyclic and the kernel of $(\partial_{\mathscr{E}})^n : \mathscr{E}^n \longrightarrow \mathscr{E}^{n+1}$ is flat for every $n \in \mathbb{Z}$.

(iii) $\mathscr{E}$ is acyclic and $\mathbb{K}$-flat.

(iv) For any complex $\mathscr{F}$ of sheaves of modules the complex $\mathscr{F} \otimes \mathscr{E}$ is acyclic.

Proof. $(i) \Rightarrow (ii)$ If $\mathscr{E}$ belongs to $\mathbb{E}(X)$ then it is acyclic, and we need to show that the quasi-coherent sheaf $Ker(\partial^n_{\mathscr{E}})$ is flat. Note that

$$\cdots \longrightarrow \mathscr{E}^{n-3} \longrightarrow \mathscr{E}^{n-2} \longrightarrow \mathscr{E}^{n-1} \longrightarrow Ker(\partial^n_{\mathscr{E}}) \longrightarrow 0$$

is a flat resolution of $Ker(\partial^n_{\mathscr{E}})$. Given a sheaf of modules $\mathscr{F}$ the complex $\mathscr{F} \otimes \mathscr{E}$ is acyclic by assumption, so the following complex is exact

$$\cdots \longrightarrow \mathscr{F} \otimes \mathscr{E}^{n-3} \longrightarrow \mathscr{F} \otimes \mathscr{E}^{n-2} \longrightarrow \mathscr{F} \otimes \mathscr{E}^{n-1}$$

This is another way of saying that the Tor sheaf $\mathscr{T}or_i(\mathscr{F}, Ker(\partial^n_{\mathscr{E}}))$ vanishes for $i > 0$. Since $\mathscr{F}$ was arbitrary, this implies that $Ker(\partial^n_{\mathscr{E}})$ is flat, as claimed.

$(ii) \Rightarrow (iii)$ Let $\mathscr{E}$ be an acyclic complex of flat quasi-coherent sheaves with flat kernels. For each $n \geq 0$ the truncation $\mathscr{E}_{\leq n}$ (see Section 2.1 for the notation) is a bounded above complex of flat quasi-coherent sheaves, hence $\mathbb{K}$-flat, from which it follows that the direct limit $\mathscr{E} = \varinjlim \mathscr{E}_{\leq n}$ is $\mathbb{K}$-flat [Lip, §2.5].

$(iii) \Rightarrow (iv)$ Given a complex $\mathscr{F}$ of sheaves of modules, we can find a quasi-isomorphism $\mathscr{P} \longrightarrow \mathscr{F}$ with $\mathscr{P}$ a $\mathbb{K}$-flat complex [Lip, Proposition 2.5.5]. Extending to a triangle, and tensoring with $\mathscr{E}$, we have a triangle in $\mathbb{K}(X)$ with $\mathscr{C}$ acyclic

$$\mathscr{P} \otimes \mathscr{E} \longrightarrow \mathscr{F} \otimes \mathscr{E} \longrightarrow \mathscr{C} \otimes \mathscr{E} \longrightarrow \Sigma(\mathscr{P} \otimes \mathscr{E})$$

Since $\mathscr{E}$ is $\mathbb{K}$-flat, $\mathscr{C} \otimes \mathscr{E}$ is acyclic. Moreover, $\mathscr{E}$ is acyclic and $\mathscr{P}$ is $\mathbb{K}$-flat, so $\mathscr{P} \otimes \mathscr{E}$ must also be acyclic. From the triangle we conclude that $\mathscr{F} \otimes \mathscr{E}$ is acyclic. Finally, $(iv) \Rightarrow (i)$ is trivial, so the proof is complete. □

Every result about $\mathbb{K}_m(\operatorname{Proj} X)$ for schemes specializes to a statement about complexes of flat modules over a commutative ring; in fact, the theory also makes sense over noncommutative rings. In the next remark we recall some features of the affine case established earlier by Neeman.

Remark 3.5. Let A be a noncommutative ring, and let $\mathbb{K}(\operatorname{Flat} A), \mathbb{K}(\operatorname{Proj} A)$ denote the homotopy categories of flat (resp. projective) left A-modules. Let $\mathbb{E}(A)$ be the triangulated subcategory of $\mathbb{K}(\operatorname{Flat} A)$ consisting of acyclic complexes with flat kernels, and define

$$\mathbb{K}_m(\operatorname{Proj} A) = \mathbb{K}(\operatorname{Flat} A)/\mathbb{E}(A)$$

In this situation the characterizations of Proposition 3.4 hold and are due to Neeman, who gives a different proof; see [Nee06a, Theorem 7.7, Corollary 8.4]. As many of our theorems rely crucially on Neeman's papers [Nee06a] and [Nee06c] we recall here some of his results:

(i) The subcategory $\mathbb{E}(A)$ is equal to the orthogonal $\mathbb{K}(\operatorname{Proj} A)^\perp$ in $\mathbb{K}(\operatorname{Flat} A)$ [Nee06a, Theorem 7.7]. Thus, a complex E of flat A-modules belongs to $\mathbb{E}(A)$ if and only if $\operatorname{Hom}_A(P, E)$ is acyclic for every complex P of projective A-modules.

(ii) The category $\mathbb{K}(\operatorname{Proj} A)$ is well generated, so by Brown-Neeman representability the inclusion $\mathbb{K}(\operatorname{Proj} A) \longrightarrow \mathbb{K}(\operatorname{Flat} A)$ has a right adjoint [Nee06a, Corollary 7.1]. The reader unfamiliar with well generated triangulated categories is referred to [Nee01b]. We use the theory to avoid a noetherian hypothesis in Theorem 5.5.

(iii) Using flat covers, Neeman proves that the inclusion $\mathbb{E}(A) \longrightarrow \mathbb{K}(\mathrm{Flat}\, A)$ has a right adjoint; see [Nee06c, Theorem 3.1]. This inclusion has a left adjoint by (*ii*) and the standard theory of Bousfield localization, so we have a recollement

$$\mathbb{E}(A) \rightleftarrows \mathbb{K}(\mathrm{Flat}\, A) \rightleftarrows \mathbb{K}_m(\mathrm{Proj}\, A)$$

There is an equivalence of $\mathbb{K}(\mathrm{Proj}\, A)$, as a subcategory of $\mathbb{K}(\mathrm{Flat}\, A)$, with the orthogonal ${}^{\perp}(\mathbb{K}(\mathrm{Proj}\, A)^{\perp}) = {}^{\perp}\mathbb{E}(A)$. From Lemma 2.3(*iv*) we conclude that the composite

$$\mathbb{K}(\mathrm{Proj}\, A) \xrightarrow{\mathrm{inc}} \mathbb{K}(\mathrm{Flat}\, A) \xrightarrow{\mathrm{can}} \mathbb{K}_m(\mathrm{Proj}\, A)$$

is an equivalence of triangulated categories.

The class $\mathbb{E}(A)$ appeared in the literature well before Neeman's [Nee06a], where it is denoted $\mathcal{S}$. In [EGR98] Enochs and Rozas call these complexes *flat*. But apart from some overlap between [Nee06a, Theorem 7.7] and [EGR98, Theorem 2.4] the two papers are very different. One can also think of the complexes in $\mathbb{E}(A)$ as the *pure exact* complexes of flat modules; see for example [Chr98, §9.1].

Over an affine scheme, the mock homotopy category is the ordinary homotopy category.

Lemma 3.6. *Let $X = Spec(A)$ be an affine scheme. There is an equivalence*

$$\mathbb{K}(\mathrm{Proj}\, A) \xrightarrow{\sim} \mathbb{K}_m(\mathrm{Proj}\, X)$$

of triangulated categories.

Proof. The equivalence $\mathbf{Mod}A \cong \mathfrak{Qco}(X)$ identifies flat A-modules with flat sheaves, and induces an equivalence of triangulated categories $\mathbb{K}(\mathrm{Flat}\, A) \cong \mathbb{K}(\mathrm{Flat}\, X)$, which becomes an equivalence $\mathbb{K}_m(\mathrm{Proj}\, A) \cong \mathbb{K}_m(\mathrm{Proj}\, X)$ of the quotients. Combining this observation with Remark 3.5(*iii*) we have an equivalence of triangulated categories

$$\mathbb{K}(\mathrm{Proj}\, A) \xrightarrow{\sim} \mathbb{K}_m(\mathrm{Proj}\, A) \xrightarrow{\sim} \mathbb{K}_m(\mathrm{Proj}\, X)$$

sending a complex of projective modules to the associated complex of flat quasi-coherent sheaves on the affine scheme. ☐

Remark 3.7. Let us given an elementary reason to care about the triangulated category $\mathbb{K}_m(\mathrm{Proj}\, A)$, as opposed to the equivalent category $\mathbb{K}(\mathrm{Proj}\, A)$. Of course, there is no formal difference, but it can be clearer to work with flat complexes rather than the associated complexes of projective modules.

For example, let A be a ring. Given an A-module M with projective resolution P and flat resolution F, there is a morphism of complexes $P \longrightarrow F$ lifting the identity

$$\begin{tikzcd}
\cdots \arrow[r] & P^{-2} \arrow[r] \arrow[d] & P^{-1} \arrow[r] \arrow[d] & P^0 \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "1"] & 0 \\
\cdots \arrow[r] & F^{-2} \arrow[r] & F^{-1} \arrow[r] & F^0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Taking the mapping cone E determines a triangle in $\mathbb{K}(\operatorname{Flat} A)$

$$P \longrightarrow F \longrightarrow E \longrightarrow \Sigma P$$

where E is an acyclic, bounded above complex of flats. Any bounded above complex of flats is $\mathbb{K}$-flat, so E belongs to $\mathbb{E}(A)$ by Proposition 3.4. We conclude that

$$P \xrightarrow{\sim} F \text{ is an isomorphism in } \mathbb{K}_m(\operatorname{Proj} A)$$

In particular, this shows that *any two flat resolutions of M are isomorphic in* $\mathbb{K}_m(\operatorname{Proj} A)$. More generally, any two $\mathbb{K}$-flat resolutions by flat quasi-coherent sheaves of a complex of quasi-coherent sheaves are isomorphic in $\mathbb{K}_m(\operatorname{Proj} X)$; see Remark 5.9 below.

Taking stalks at a point $x \in X$ and restricting to an open subset $U \subseteq X$ both preserve flatness, so we have coproduct preserving triangulated functors

$$(-)|_U : \mathbb{K}(\operatorname{Flat} X) \longrightarrow \mathbb{K}(\operatorname{Flat} U) \tag{3.1}$$

$$(-)_x : \mathbb{K}(\operatorname{Flat} X) \longrightarrow \mathbb{K}(\operatorname{Flat} \mathcal{O}_{X,x}) \tag{3.2}$$

Let $f : U \longrightarrow X$ be the inclusion of an affine open subset. Then $f_* : \mathfrak{Qco}(U) \longrightarrow \mathfrak{Qco}(X)$ sends flat sheaves to flat sheaves, and there is a triangulated functor

$$f_* : \mathbb{K}(\operatorname{Flat} U) \longrightarrow \mathbb{K}(\operatorname{Flat} X) \tag{3.3}$$

which, by a standard argument, preserves coproducts and is right adjoint to (3.1).

Lemma 3.8. *Let $\mathscr{E}$ be a complex of flat quasi-coherent sheaves. Then $\mathscr{E} \in \mathbb{E}(X)$ if and only if $\mathscr{E}_x \in \mathbb{E}(\mathcal{O}_{X,x})$ for every $x \in X$. It follows that*

(i) If $U \subseteq X$ is an open subset and $\mathscr{E} \in \mathbb{E}(X)$ then $\mathscr{E}|_U \in \mathbb{E}(U)$.

(ii) If $\{V_i\}_{i \in I}$ is an open cover of X then $\mathscr{E} \in \mathbb{E}(X)$ if and only if $\mathscr{E}|_{V_i} \in \mathbb{E}(V_i)$ for all $i \in I$.

Proof. We know from Proposition 3.4 that $\mathscr{E}$ belongs to $\mathbb{E}(X)$ if and only if it is acyclic with flat kernels, both of which can be checked on stalks, so the claims are immediate. □

Definition 3.9. Taking stalks at a point $x \in X$ sends $\mathbb{E}(X)$ into $\mathbb{E}(\mathcal{O}_{X,x})$, while restricting to an open subset $U \subseteq X$ sends $\mathbb{E}(X)$ into $\mathbb{E}(U)$, so the functors of (3.1), (3.2) induce coproduct preserving triangulated functors on the quotients

$$(-)|_U : \mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{K}_m(\operatorname{Proj} U) \tag{3.4}$$

$$(-)_x : \mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{K}_m(\operatorname{Proj} \mathcal{O}_{X,x}) \tag{3.5}$$

Let $f : U \longrightarrow X$ be the inclusion of an affine open subset. Then $f_* : \mathfrak{Qco}(U) \longrightarrow \mathfrak{Qco}(X)$ is exact and sends flat sheaves to flat sheaves, and using characterization (ii) of Proposition 3.4 we infer that $f_*\mathbb{E}(U) \subseteq \mathbb{E}(X)$. It follows that (3.3) induces a triangulated functor

$$f_* : \mathbb{K}_m(\operatorname{Proj} U) \longrightarrow \mathbb{K}_m(\operatorname{Proj} X) \tag{3.6}$$

which, by a standard argument, is coproduct preserving and right adjoint to (3.4).

Given an additive category $\mathcal{A}$, an object X *is zero in* $\mathcal{A}$ (or just, *is zero*, when there is no chance of confusion) when $\mathrm{Hom}_{\mathcal{A}}(X, X) = 0$. We are often dealing with complexes, so it is worth being clear: a complex X of objects in $\mathcal{A}$ is zero in the homotopy category $\mathbb{K}(\mathcal{A})$ if and only if it is contractible.

Remark 3.10. A complex $\mathscr{F}$ of flat quasi-coherent sheaves is zero in $\mathbb{K}_m(\mathrm{Proj}\, X)$ if and only if it belongs to $\mathbb{E}(X)$,[1] which by Lemma 3.8 is a local question. That is,

(i) $\mathscr{F} = 0$ in $\mathbb{K}_m(\mathrm{Proj}\, X)$ if and only if $\mathscr{F}_x = 0$ in $\mathbb{K}_m(\mathrm{Proj}\, \mathcal{O}_{X,x})$ for every $x \in X$.

(ii) If $\{V_i\}_{i \in I}$ is an open cover of X then $\mathscr{F} = 0$ in $\mathbb{K}_m(\mathrm{Proj}\, X)$ if and only if $\mathscr{F}|_{V_i} = 0$ in $\mathbb{K}_m(\mathrm{Proj}\, V_i)$ for every $i \in I$.

Being an isomorphism is also local, because a morphism $f : \mathscr{F} \longrightarrow \mathscr{G}$ is an isomorphism in $\mathbb{K}_m(\mathrm{Proj}\, X)$ if and only if the mapping cone in $\mathbb{K}_m(\mathrm{Proj}\, X)$ is zero.

The triangles in $\mathbb{K}_m(\mathrm{Proj}\, X)$ are the candidate triangles isomorphic, in $\mathbb{K}_m(\mathrm{Proj}\, X)$, to a triangle from $\mathbb{K}(\mathrm{Flat}\, X)$ [Nee01b, §2.1]. Isomorphism in $\mathbb{K}_m(\mathrm{Proj}\, X)$ is weaker than homotopy equivalence, so there are triangles in the mock homotopy category not apparent in the homotopy category of flat sheaves. For example, suppose we have an exact sequence of complexes of flat quasi-coherent sheaves

$$0 \longrightarrow \mathscr{A} \xrightarrow{\varphi} \mathscr{B} \xrightarrow{\psi} \mathscr{C} \longrightarrow 0 \tag{3.7}$$

It is standard that there is a triangle $\mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{C} \longrightarrow \Sigma\mathscr{A}$ in the derived category $\mathbb{D}(\mathfrak{Qco} X)$. The next lemma observes that this triangle exists already in $\mathbb{K}_m(\mathrm{Proj}\, X)$.

Note that a short exact sequence of complexes of projective or injective modules must split in each degree and determine a triangle in $\mathbb{K}(\mathrm{Proj}\, A)$ or $\mathbb{K}(\mathrm{Inj}\, A)$, by Lemma 2.15. Flats do not split sequences, but they do determine triangles.

Lemma 3.11. *Given an exact sequence* (3.7) *of complexes of flat quasi-coherent sheaves, there is a canonical morphism* $z : \mathscr{C} \longrightarrow \Sigma\mathscr{A}$ *and triangle, both in* $\mathbb{K}_m(\mathrm{Proj}\, X)$

$$\mathscr{A} \xrightarrow{\varphi} \mathscr{B} \xrightarrow{\psi} \mathscr{C} \xrightarrow{-z} \Sigma\mathscr{A} \tag{3.8}$$

Proof. The morphism ψ factorizes as $\mathscr{B} \longrightarrow \mathrm{cone}(\varphi) \xrightarrow{\mu} \mathscr{C}$ where $\mathrm{cone}(\varphi)$ is the mapping cone and μ is a quasi-isomorphism, by a standard argument of homological algebra. We are claiming that μ is actually an isomorphism in $\mathbb{K}_m(\mathrm{Proj}\, X)$. That is, if we claim that if you extend μ to a triangle in $\mathbb{K}(\mathrm{Flat}\, X)$

$$\mathrm{cone}(\varphi) \xrightarrow{\mu} \mathscr{C} \longrightarrow \mathscr{E} \longrightarrow \Sigma\,\mathrm{cone}(\varphi) \tag{3.9}$$

then $\mathscr{E}$ belongs to $\mathbb{E}(X)$. In each degree (3.7) is exact and consists of flat sheaves, so it is degree-wise exact and $\mathbb{K}$-flat, and remains exact after tensoring with any sheaf. That is, for any sheaf of modules $\mathscr{F}$ we have a short exact sequence of complexes of sheaves

$$0 \longrightarrow \mathscr{F} \otimes \mathscr{A} \longrightarrow \mathscr{F} \otimes \mathscr{B} \longrightarrow \mathscr{F} \otimes \mathscr{C} \longrightarrow 0$$

[1] Here we actually use thickness of $\mathbb{E}(X)$, see [Nee01b, Lemma 2.1.33].

and a quasi-isomorphism $\mathrm{cone}(\mathscr{F} \otimes \varphi) \longrightarrow \mathscr{F} \otimes \mathscr{C}$. Tensoring with $\mathscr{F}$ is a triangulated functor, so $\mathrm{cone}(\mathscr{F} \otimes \varphi) \cong \mathscr{F} \otimes \mathrm{cone}(\varphi)$ and $\mathscr{F} \otimes \mu$ is a quasi-isomorphism for every sheaf of modules $\mathscr{F}$. From the triangle (3.9) we conclude that $\mathscr{F} \otimes \mathscr{E}$ is acyclic, whence $\mathscr{E}$ belongs to $\mathbb{E}(X)$, as claimed. Let $w : \mathrm{cone}(\varphi) \longrightarrow \Sigma\mathscr{A}$ denote the canonical morphism of complexes out of the mapping cone, and set $z = -w \circ \mu^{-1}$ in $\mathbb{K}_m(\mathrm{Proj}\, X)$. The candidate triangle (3.8) is isomorphic in $\mathbb{K}_m(\mathrm{Proj}\, X)$ to the mapping cone triangle $\mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathrm{cone}(\varphi) \longrightarrow \Sigma\mathscr{A}$, and is therefore itself a triangle. □

3.1 Čech Triangles

In this section we construct a sequence of *Čech triangles* in $\mathbb{K}_m(\mathrm{Proj}\, X)$ that assemble a complex of flat quasi-coherent sheaves from its restrictions to an open affine cover. This will allow us to prove global statements about $\mathbb{K}_m(\mathrm{Proj}\, X)$ using local arguments.

Setup. In this section X is a scheme with affine open cover $\mathfrak{U} = \{U_0, \ldots, U_d\}$ and sheaves are defined over X by default.

Given a quasi-coherent sheaf $\mathscr{F}$ we have an exact sequence of quasi-coherent sheaves called the *Čech resolution* which is, in the notation of [Har77, §III.4]

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \cdots \longrightarrow \mathscr{C}^d(\mathfrak{U}, \mathscr{F}) \longrightarrow 0$$
$$\mathscr{C}^p(\mathfrak{U}, \mathscr{F}) = \oplus_{i_0 < \cdots < i_p} f_*(\mathscr{F}|_{U_{i_0,\ldots,i_p}})$$

where the *Čech sheaf* $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is a direct sum over sequences $i_0 < \cdots < i_p$ of length p in $\{0, \ldots, d\}$ and $f : U_{i_0,\ldots,i_p} \longrightarrow X$ is the inclusion of the open set $U_{i_0,\ldots,i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$. Taking the pth Čech sheaf defines an exact coproduct preserving functor

$$\mathscr{C}^p(\mathfrak{U}, -) : \mathfrak{Qco}(X) \longrightarrow \mathfrak{Qco}(X)$$

For a *complex* of quasi-coherent sheaves $\mathscr{F}$ we can take Čech resolutions of each term to obtain a bicomplex of quasi-coherent sheaves (see Section 2.1 for background)

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
\cdots \arrow[r] & \mathscr{C}^d(\mathfrak{U}, \mathscr{F}^{n-1}) \arrow[r] \arrow[u] & \mathscr{C}^d(\mathfrak{U}, \mathscr{F}^{n}) \arrow[r] \arrow[u] & \mathscr{C}^d(\mathfrak{U}, \mathscr{F}^{n+1}) \arrow[r] \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] & \\
\cdots \arrow[r] & \mathscr{C}^0(\mathfrak{U}, \mathscr{F}^{n-1}) \arrow[r] \arrow[u] & \mathscr{C}^0(\mathfrak{U}, \mathscr{F}^{n}) \arrow[r] \arrow[u] & \mathscr{C}^0(\mathfrak{U}, \mathscr{F}^{n+1}) \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & \mathscr{F}^{n-1} \arrow[r] \arrow[u] & \mathscr{F}^{n} \arrow[r] \arrow[u] & \mathscr{F}^{n+1} \arrow[r] \arrow[u] & \cdots \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd} \tag{3.10}$$

Denote by $\mathscr{C}(\mathfrak{U},\mathscr{F})$ the upper part of this bicomplex formed by just the Čech sheaves (that is, delete the bottom row containing $\mathscr{F}$). In column $i \in \mathbb{Z}$ and row $j \geq 0$ the sheaf in this bicomplex is $\mathscr{C}(\mathfrak{U},\mathscr{F})^{ij} = \mathscr{C}^j(\mathfrak{U},\mathscr{F}^i)$. The next lemma tells us that in $\mathbb{K}_m(\operatorname{Proj} X)$ the complex $\mathscr{F}$ is isomorphic to the totalization of this bicomplex.

Lemma 3.12. *Given a complex $\mathscr{F}$ of flat quasi-coherent sheaves there is a triangle in $\mathbb{K}(\operatorname{Flat} X)$ with $\mathscr{E}$ an object of $\mathbb{E}(X)$*

$$\mathscr{F} \longrightarrow \operatorname{Tot}\mathscr{C}(\mathfrak{U},\mathscr{F}) \longrightarrow \mathscr{E} \longrightarrow \Sigma\mathscr{F}$$

In particular, there is an isomorphism $\mathscr{F} \xrightarrow{\sim} \operatorname{Tot}\mathscr{C}(\mathfrak{U},\mathscr{F})$ in $\mathbb{K}_m(\operatorname{Proj} X)$.

Proof. Note that $\mathscr{C}(\mathfrak{U},\mathscr{F})$ is a bicomplex of *flat* quasi-coherent sheaves because each Čech sheaf $\mathscr{C}^j(\mathfrak{U},\mathscr{F}^i)$ is a coproduct of direct images of flat sheaves under a flat affine morphism, and such direct image sheaves are flat.

Let D denote the bicomplex in (3.10) which has the bicomplex $\mathscr{C}(\mathfrak{U},\mathscr{F})$ in rows ≥ 0 and the complex $\mathscr{F}$ in row -1. Actually, to get the signs right in what follows we should take D to be the bicomplex in (3.10) but with some signs modified, as indicated in the following diagram

$$\begin{tikzcd} \cdots \arrow[r, "+"] & \mathscr{C}^0(\mathfrak{U},\mathscr{F}^{-1}) \arrow[r, "+"] & \mathscr{C}^0(\mathfrak{U},\mathscr{F}^{0}) \arrow[r, "+"] & \mathscr{C}^1(\mathfrak{U},\mathscr{F}^{1}) \arrow[r, "+"] & \cdots \\ \cdots \arrow[r, "-"'] & \mathscr{F}^{-1} \arrow[u, "-"] \arrow[r, "-"'] & \mathscr{F}^{0} \arrow[u, "+"] \arrow[r, "-"'] & \mathscr{F}^{1} \arrow[u, "-"] \arrow[r, "-"'] & \cdots \end{tikzcd} \tag{3.11}$$

The -1st row $\mathscr{F}$ of (3.10) gets negative signs on its morphisms in D, and we alternate the signs on the morphisms between the $\mathscr{F}^i$ and $\mathscr{C}^0(\mathfrak{U},\mathscr{F}^i)$, as indicated. Everywhere else the signs on the morphisms in D agree with the natural ones in (3.10).

There is a canonical morphism of bicomplexes $\mathscr{C}(\mathfrak{U},\mathscr{F}) \longrightarrow D$, whose cokernel is a bicomplex nonzero only in row -1, where it is $\mathscr{F}$ with a negative sign on all its differentials, as in (3.11). This gives rise to a short exact sequence of bicomplexes, split exact in each bidegree, and then by Lemma 2.17 we obtain a triangle in $\mathbb{K}(\operatorname{Flat} X)$

$$\operatorname{Tot}\mathscr{C}(\mathfrak{U},\mathscr{F}) \longrightarrow \operatorname{Tot}(D) \longrightarrow \Sigma\mathscr{F} \xrightarrow{-\Sigma u} \Sigma\operatorname{Tot}\mathscr{C}(\mathfrak{U},\mathscr{F})$$

where $u^n : \mathscr{F}^n \longrightarrow \operatorname{Tot}\mathscr{C}^0(\mathfrak{U},\mathscr{F})^n$ is the first morphism $\mathscr{F}^n \longrightarrow \mathscr{C}^0(\mathfrak{U},\mathscr{F}^n)$ in the Čech resolution composed with the inclusion of $\mathscr{C}^0(\mathfrak{U},\mathscr{F}^n)$ into the coproduct $\oplus_{j\geq 0}\mathscr{C}^j(\mathfrak{U},\mathscr{F}^{n-j})$. Shifting, we have

$$\mathscr{F} \xrightarrow{u} \operatorname{Tot}\mathscr{C}(\mathfrak{U},\mathscr{F}) \longrightarrow \operatorname{Tot}(D) \longrightarrow \Sigma\mathscr{F} \tag{3.12}$$

It is well-known that the bicomplex D_x of $\mathcal{O}_{X,x}$-modules has contractible columns [Har77, III.4.2] so Lemma 2.19 implies that the totalization $\operatorname{Tot}(D_x) \cong \operatorname{Tot}(D)_x$ is contractible, and therefore in the subcategory $\mathbb{E}(\mathcal{O}_{X,x})$, for every $x \in X$. Using Lemma 3.8 we conclude that $\operatorname{Tot}(D) \in \mathbb{E}(X)$, so $u : \mathscr{F} \longrightarrow \operatorname{Tot}\mathscr{C}(\mathfrak{U},\mathscr{F})$ is an isomorphism in $\mathbb{K}_m(\operatorname{Proj} X)$. $\square$

Now we are ready to define the Čech triangles. Given a bicomplex B, we denote by $B_{\mathrm{rows}\geq k}$ the *brutal row truncation* of B which deletes the rows $< k$. Recall also that our scheme X comes with an affine open cover $\mathfrak{U}$ with $d+1$ elements.

Proposition 3.13. *Associated to every complex $\mathscr{F}$ of flat quasi-coherent sheaves is a canonical sequence of triangles in $\mathbb{K}_m(\operatorname{Proj} X)$*

$$\begin{gathered}
\mathscr{P}_{d-1} \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U},\mathscr{F}) \longrightarrow \Sigma\mathscr{P}_{d-1}\\
\mathscr{P}_{d-2} \longrightarrow \mathscr{P}_{d-1} \longrightarrow \Sigma^{-1}\mathscr{C}^1(\mathfrak{U},\mathscr{F}) \longrightarrow \Sigma\mathscr{P}_{d-2}\\
\vdots\\
\mathscr{P}_{i} \longrightarrow \mathscr{P}_{i+1} \longrightarrow \Sigma^{-d+i+1}\mathscr{C}^{d-i-1}(\mathfrak{U},\mathscr{F}) \longrightarrow \Sigma\mathscr{P}_{i}\\
\vdots\\
\mathscr{P}_{1} \longrightarrow \mathscr{P}_{2} \longrightarrow \Sigma^{-d+2}\mathscr{C}^{d-2}(\mathfrak{U},\mathscr{F}) \longrightarrow \Sigma\mathscr{P}_{1}\\
\Sigma^{-d}\mathscr{C}^{d}(\mathfrak{U},\mathscr{F}) \longrightarrow \mathscr{P}_{1} \longrightarrow \Sigma^{-d+1}\mathscr{C}^{d-1}(\mathfrak{U},\mathscr{F}) \longrightarrow \Sigma^{1-d}\mathscr{C}^{d}(\mathfrak{U},\mathscr{F})
\end{gathered}$$

where $\mathscr{P}_i = \operatorname{Tot}(\mathscr{C}(\mathfrak{U},\mathscr{F})_{\mathrm{rows}\geq d-i})$. Moreover, these triangles are natural in $\mathscr{F}$.

Proof. Associated to the bicomplex $B = \mathscr{C}(\mathfrak{U},\mathscr{F})$ are the brutal row truncations $B_{\mathrm{rows}\geq d-i}$ and their totalizations $\mathscr{P}_i = \operatorname{Tot}(B_{\mathrm{rows}\geq d-i})$. For $i \geq 0$ there are canonical triangles in $\mathbb{K}(\operatorname{Flat} X)$ that relate successive truncations (see the triangle (2.14) in Section 2.1)

$$\mathscr{P}_{i} \longrightarrow \mathscr{P}_{i+1} \longrightarrow \Sigma^{-d+i+1}\mathscr{C}^{d-i-1}(\mathfrak{U},\mathscr{F}) \longrightarrow \Sigma\mathscr{P}_{i} \tag{3.13}$$

By inspection we have $\mathscr{P}_0 = \Sigma^{-d}\mathscr{C}^d(\mathfrak{U},\mathscr{F})$ and $\mathscr{P}_d = \operatorname{Tot}\mathscr{C}(\mathfrak{U},\mathscr{F})$. This yields a sequence of triangles in $\mathbb{K}(\operatorname{Flat} X)$, and therefore $\mathbb{K}_m(\operatorname{Proj} X)$, accounting for all the triangles in the statement of the proposition except for the first. From (3.13) we have a triangle

$$\mathscr{P}_{d-1} \longrightarrow \operatorname{Tot}\mathscr{C}(\mathfrak{U},\mathscr{F}) \longrightarrow \mathscr{C}^0(\mathfrak{U},\mathscr{F}) \longrightarrow \Sigma\mathscr{P}_{d-1} \tag{3.14}$$

In $\mathbb{K}_m(\operatorname{Proj} X)$ we can replace $\operatorname{Tot}\mathscr{C}(\mathfrak{U},\mathscr{F})$ by the isomorphic object $\mathscr{F}$, which completes the construction of the Čech triangles. It only remains to discuss naturality.

Given a morphism $\phi : \mathscr{F} \longrightarrow \mathscr{G}$ of complexes of flat quasi-coherent sheaves there is an induced morphism of bicomplexes $\mathscr{C}(\mathfrak{U},\mathscr{F}) \longrightarrow \mathscr{C}(\mathfrak{U},\mathscr{G})$ which truncates to a morphism of complexes $\mathscr{P}_i \longrightarrow \mathscr{Q}_i$ for each $i \geq 0$, where $\mathscr{Q}_i = \operatorname{Tot}(\mathscr{C}(\mathfrak{U},\mathscr{G})_{\mathrm{rows}\geq d-i})$. We deduce a morphism of triangles in $\mathbb{K}(\operatorname{Flat} X)$ for $i \geq 0$

$$\begin{tikzcd}
\mathscr{P}_{i} \arrow[r] \arrow[d] & \mathscr{P}_{i+1} \arrow[r] \arrow[d] & \Sigma^{-d+i+1}\mathscr{C}^{d-i-1}(\mathfrak{U},\mathscr{F}) \arrow[r] \arrow[d] & \Sigma\mathscr{P}_{i} \arrow[d]\\
\mathscr{Q}_{i} \arrow[r] & \mathscr{Q}_{i+1} \arrow[r] & \Sigma^{-d+i+1}\mathscr{C}^{d-i-1}(\mathfrak{U},\mathscr{G}) \arrow[r] & \Sigma\mathscr{Q}_{i}
\end{tikzcd}$$

When $i = d-1$ we use naturality of the isomorphism $\mathscr{F} \longrightarrow \operatorname{Tot}\mathscr{C}(\mathfrak{U},\mathscr{F})$ of Lemma 3.12 to check that the following diagram commutes in $\mathbb{K}_m(\operatorname{Proj} X)$

$$\begin{tikzcd}
\mathscr{P}_{d-1} \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d, "\phi"] & \mathscr{C}^0(\mathfrak{U},\mathscr{F}) \arrow[r] \arrow[d] & \Sigma\mathscr{P}_{d-1} \arrow[d]\\
\mathscr{Q}_{d-1} \arrow[r] & \mathscr{G} \arrow[r] & \mathscr{C}^0(\mathfrak{U},\mathscr{G}) \arrow[r] & \Sigma\mathscr{Q}_{d-1}
\end{tikzcd}$$

which proves naturality of the Čech triangles with respect to morphisms in $\mathbb{K}(\operatorname{Flat} X)$. $\square$

We give three applications of the Čech triangles, the first of which says that $\mathbb{K}_m(\operatorname{Proj} X)$ is generated by complexes defined over its affine open subsets. This is the form in which the previous result will usually be applied.

Corollary 3.14. *Let $\mathcal{L}$ be a triangulated subcategory of $\mathbb{K}_m(\operatorname{Proj} X)$ and $\mathscr{F}$ a complex of flat quasi-coherent sheaves. Suppose that for any intersection $V = U_{i_0} \cap \cdots \cap U_{i_p}$ of open sets in the cover $\mathfrak{U}$ we have $f_*(\mathscr{F}|_V) \in \mathcal{L}$, where $f : V \longrightarrow X$ denotes the inclusion. Then $\mathscr{F}$ belongs to $\mathcal{L}$.*

Proof. Taking finite direct sums we deduce that the Čech complex $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ in degree p belongs to $\mathcal{L}$ for every $0 \le p \le d$. The last Čech triangle of Proposition 3.13 has the form

$$\Sigma^{-d}\mathscr{C}^d(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{P}_1 \longrightarrow \Sigma^{-d+1}\mathscr{C}^{d-1}(\mathfrak{U}, \mathscr{F}) \longrightarrow \Sigma^{1-d}\mathscr{C}^d(\mathfrak{U}, \mathscr{F})$$

Because the Čech complexes belong to $\mathcal{L}$ we deduce that $\mathscr{P}_1$ belongs to $\mathcal{L}$. Climbing up the sequence of Čech triangles, we find that every $\mathscr{P}_i$ belongs to $\mathcal{L}$. Finally, from

$$\mathscr{P}_{d-1} \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \Sigma\mathscr{P}_{d-1}$$

we conclude that $\mathscr{F}$ belongs to $\mathcal{L}$, as required. $\square$

Lemma 3.15. *Given a cover $\{V_\alpha\}_{\alpha \in \Lambda}$ of X by quasi-compact open subsets, an object $\mathscr{X}$ is compact in $\mathbb{K}_m(\operatorname{Proj} X)$ if and only if $\mathscr{X}|_{V_\alpha}$ is compact in $\mathbb{K}_m(\operatorname{Proj} V_\alpha)$ for every $\alpha \in \Lambda$.*

Proof. First we prove that a locally compact object is globally compact. That is, given a complex $\mathscr{X}$ of flat quasi-coherent sheaves with $\mathscr{X}|_{V_\alpha}$ compact in $\mathbb{K}_m(\operatorname{Proj} V_\alpha)$ for every $\alpha \in \Lambda$, we have to prove that $\operatorname{Hom}(\mathscr{X}, -)$ preserves coproducts (to reduce clutter we omit subscripts on Homs. It is understood that all Homs are taken in $\mathbb{K}_m(\operatorname{Proj} -)$ over an open subset of X that will be clear from the context). The first step in the proof is the following observation:

> ($\star$) Let $f : U \longrightarrow X$ be the inclusion of an affine open subset, so that we know, by Definition 3.9, that $(-)|_U : \mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{K}_m(\operatorname{Proj} U)$ has a coproduct preserving right adjoint and therefore preserves compactness by Lemma 2.9.

Let $\mathfrak{U} = \{U_0, \ldots, U_d\}$ be an affine open cover of X with each U_i contained in some V_α. Every finite intersection U of open sets in the cover $\mathfrak{U}$ is an affine open subset of some V_α, so from the hypothesis and ($\star$) we infer that $\mathscr{X}|_U$ is compact. We prove that $\operatorname{Hom}(\mathscr{X}, -)$ preserves coproducts by using Čech triangles to reduce the problem to compactness over such finite intersections.

Suppose that we are given a family $\{\mathscr{F}_s\}_{s \in T}$ of complexes of flat quasi-coherent sheaves and let $\oplus_s \mathscr{F}_s$ be the coproduct in $\mathbb{K}(\operatorname{Flat} X)$. The Čech triangles of Proposition 3.13 assemble each complex $\mathscr{F}_s$, as well as the coproduct $\oplus_s \mathscr{F}_s$, from their restrictions to affine open subsets, and the inclusion $\mathscr{F}_s \longrightarrow \oplus_s \mathscr{F}_s$ induces morphisms of these triangles. More precisely, for each $s \in T$ and $0 \le i \le d$ we set

$$\mathscr{P}_{s,i} = \mathrm{Tot}(\mathscr{C}(\mathfrak{U}, \mathscr{F}_s)_{\mathrm{rows} \ge d-i}), \quad \mathscr{P}_i = \mathrm{Tot}(\mathscr{C}(\mathfrak{U}, \oplus_s \mathscr{F}_s)_{\mathrm{rows} \ge d-i}) \tag{3.15}$$

We are working in $\mathbb{K}_m(\mathrm{Proj}\, X)$, so by Lemma 3.12 we can assume that $\mathscr{P}_{s,d} = \mathscr{F}_s$ and $\mathscr{P}_d = \oplus_s \mathscr{F}_s$. The totalizations (3.15) fit into a morphism of triangles in $\mathbb{K}_m(\mathrm{Proj}\, X)$ for every $s \in T$ and $0 \le i \le d-1$

$$\begin{tikzcd}
\mathscr{P}_{s,i} \arrow[r] \arrow[d] & \mathscr{P}_{s,i+1} \arrow[r] \arrow[d] & \Sigma^{-d+i+1}\mathscr{C}^{d-i-1}(\mathfrak{U}, \mathscr{F}_s) \arrow[r] \arrow[d] & \Sigma \mathscr{P}_{s,i} \arrow[d] \\
\mathscr{P}_i \arrow[r] & \mathscr{P}_{i+1} \arrow[r] & \Sigma^{-d+i+1}\mathscr{C}^{d-i-1}(\mathfrak{U}, \oplus_s \mathscr{F}_s) \arrow[r] & \Sigma \mathscr{P}_i
\end{tikzcd} \tag{3.16}$$

Applying $\mathrm{Hom}(\mathscr{X}, -)$ yields a morphism of long exact sequences for every $s \in T$ and $0 \le i \le d-1$. Fixing i and taking the coproduct of the top row (of this morphism of long exact sequences) over all $s \in T$ produces a morphism of long exact sequences of which the following diagram is an excerpt

$$\begin{tikzcd}
\cdots \arrow[r] & \oplus_s \mathrm{Hom}(\mathscr{X}, \mathscr{P}_{s,i+1}) \arrow[r] \arrow[d] & \oplus_s \mathrm{Hom}(\mathscr{X}, \Sigma^{-d+i+1}\mathscr{C}^{d-i-1}(\mathfrak{U}, \mathscr{F}_s)) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}(\mathscr{X}, \mathscr{P}_{i+1}) \arrow[r] & \mathrm{Hom}(\mathscr{X}, \Sigma^{-d+i+1}\mathscr{C}^{d-i-1}(\mathfrak{U}, \oplus_s \mathscr{F}_s)) \arrow[r] & \cdots
\end{tikzcd} \tag{3.17}$$

In this diagram every third vertical morphism is an isomorphism of the following type

$$\begin{aligned}
\mathrm{Hom}(\mathscr{X}, \mathscr{C}^p(\mathfrak{U}, \oplus_s \mathscr{F}_s)) &= \mathrm{Hom}(\mathscr{X}, \oplus_{i_0 < \cdots < i_p} f_*(\oplus_s \mathscr{F}_s|_{U_{i_0,\ldots,i_p}})) \\
&\cong \oplus_{i_0 < \cdots < i_p} \mathrm{Hom}(\mathscr{X}, f_*(\oplus_s \mathscr{F}_s|_{U_{i_0,\ldots,i_p}})) \\
&\cong \oplus_{i_0 < \cdots < i_p} \mathrm{Hom}(\mathscr{X}|_{U_{i_0,\ldots,i_p}}, \oplus_s \mathscr{F}_s|_{U_{i_0,\ldots,i_p}}) && \text{(Adjunction)} \\
&\cong \oplus_{i_0 < \cdots < i_p} \oplus_s \mathrm{Hom}(\mathscr{X}|_{U_{i_0,\ldots,i_p}}, \mathscr{F}_s|_{U_{i_0,\ldots,i_p}}) && \text{(Locally compact)} \\
&\cong \oplus_{i_0 < \cdots < i_p} \oplus_s \mathrm{Hom}(\mathscr{X}, f_*(\mathscr{F}_s|_{U_{i_0,\ldots,i_p}})) && \text{(Adjunction)} \\
&\cong \oplus_s \mathrm{Hom}(\mathscr{X}, \oplus_{i_0 < \cdots < i_p} f_*(\mathscr{F}_s|_{U_{i_0,\ldots,i_p}})) \\
&= \oplus_s \mathrm{Hom}(\mathscr{X}, \mathscr{C}^p(\mathfrak{U}, \mathscr{F}_s))
\end{aligned}$$

We proceed by inductively climbing the sequence of Čech triangles, beginning with the last one (in the order listed in Proposition 3.13) which corresponds to $i = 0$. With this value of i, an isomorphism of the above type occurs in two out of every three columns of (3.17) because $\mathscr{P}_{s,0} = \Sigma^{-d}\mathscr{C}^d(\mathfrak{U}, \mathscr{F}_s)$. From the Five Lemma we deduce an isomorphism

$$\oplus_s \mathrm{Hom}(\mathscr{X}, \mathscr{P}_{s,1}) \xrightarrow{\sim} \mathrm{Hom}(\mathscr{X}, \mathscr{P}_1) \tag{3.18}$$

Next, for $i = 1$, every third column of (3.17) is once again an isomorphism and, using (3.18) and the Five Lemma, we conclude that every column is an isomorphism. Proceeding in this way we eventually reach the final Čech triangle ($i = d-1$), and from it we infer that there is an isomorphism

$$\oplus_s \mathrm{Hom}(\mathscr{X}, \mathscr{F}_s) \xrightarrow{\sim} \mathrm{Hom}(\mathscr{X}, \oplus_s \mathscr{F}_s) \tag{3.19}$$

which proves that $\mathscr{X}$ is compact, as required. It remains to check that when $\mathscr{X}$ is compact the restrictions $\mathscr{X}|_{V_\alpha}$ are all compact. More generally, let $U \subseteq X$ be a quasi-compact open subset and $\mathfrak{U} = \{W_0, \ldots, W_d\}$ an affine open cover of U. By $(\star)$ the restrictions $\mathscr{X}|_{W_i}$ are all compact; applying the first part of the proof to the scheme U, cover $\mathfrak{U}$ and object $\mathscr{X}|_U$ we conclude that $\mathscr{X}|_U$ is compact. □

The next theorem generalizes a result of Neeman to schemes; see Remark 3.5(*iii*).

Theorem 3.16. *There is a localization sequence*

$$\mathbb{E}(X) \rightleftarrows \mathbb{K}(\operatorname{Flat} X) \rightleftarrows \mathbb{K}_m(\operatorname{Proj} X) \tag{3.20}$$

In particular $\mathbb{K}_m(\operatorname{Proj} X)$ *has small Homs.*

Proof. The existence of a localization sequence (3.20) is equivalent by Lemma 2.3 to the existence, for every complex $\mathscr{F}$ in $\mathbb{K}(\operatorname{Flat} X)$, of a triangle in $\mathbb{K}(\operatorname{Flat} X)$ with $\mathscr{E}$ in $\mathbb{E}(X)$ and $\mathscr{Y}$ in the orthogonal $\mathbb{E}(X)^\perp$

$$\mathscr{E} \longrightarrow \mathscr{F} \longrightarrow \mathscr{Y} \longrightarrow \Sigma\mathscr{E} \tag{3.21}$$

Let $\mathcal{L}$ denote the full subcategory of $\mathbb{K}(\operatorname{Flat} X)$ consisting of the complexes $\mathscr{F}$ that fit into such a triangle; this is a triangulated subcategory, the Verdier sum of $\mathbb{E}(X)$ and $\mathbb{E}(X)^\perp$. The proof will be complete if we can show that every complex $\mathscr{F}$ in $\mathbb{K}(\operatorname{Flat} X)$ belongs to $\mathcal{L}$. By Lemma 3.12 we have a triangle in $\mathbb{K}(\operatorname{Flat} X)$

$$\mathscr{F} \longrightarrow \operatorname{Tot}\mathscr{C}(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{E} \longrightarrow \Sigma\mathscr{F} \tag{3.22}$$

with $\mathscr{E}$ in $\mathbb{E}(X)$. Taking $\mathscr{Y} = 0$ in (3.21) demonstrates that $\mathbb{E}(X) \subseteq \mathcal{L}$, so proving that $\mathscr{F}$ belongs to $\mathcal{L}$ is equivalent to proving that $\operatorname{Tot}\mathscr{C}(\mathfrak{U}, \mathscr{F})$ belongs to $\mathcal{L}$. The bicomplex $\mathscr{C}(\mathfrak{U}, \mathscr{F})$ is bounded vertically, so to show that the totalization belongs to $\mathcal{L}$ it suffices to argue that the rows $\mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ of the bicomplex all belong to $\mathcal{L}$ (Remark 2.18).

Each row is a finite direct sum $\mathscr{C}^p(\mathfrak{U}, \mathscr{F}) = \oplus_{i_0 < \cdots < i_p} f_*(\mathscr{F}|_{U_{i_0,\ldots,i_p}})$ so to complete the proof we need to show that $f_*(\mathscr{T})$ is in $\mathcal{L}$ whenever $f : V \longrightarrow X$ is the inclusion of an affine open subset and $\mathscr{T}$ is a complex of flat quasi-coherent sheaves on V. The scheme V is affine, so by [Nee06c, Theorem 3.1] we have a triangle in $\mathbb{K}(\operatorname{Flat} V)$ with $\mathscr{E}$ in $\mathbb{E}(V)$ and $\mathscr{Y}$ in the orthogonal $\mathbb{E}(V)^\perp$ (note that Neeman writes $\mathcal{S}$ for what we call $\mathbb{E}(V)$)

$$\mathscr{E} \longrightarrow \mathscr{T} \longrightarrow \mathscr{Y} \longrightarrow \Sigma\mathscr{E}$$

Applying the functor $f_* : \mathbb{K}(\operatorname{Flat} V) \longrightarrow \mathbb{K}(\operatorname{Flat} X)$ we have a triangle in $\mathbb{K}(\operatorname{Flat} X)$

$$f_*(\mathscr{E}) \longrightarrow f_*(\mathscr{T}) \longrightarrow f_*(\mathscr{Y}) \longrightarrow \Sigma f_*(\mathscr{E})$$

where $f_*(\mathscr{E})$ belongs to $\mathbb{E}(X)$ and $f_*(\mathscr{Y})$ belongs to the orthogonal $\mathbb{E}(X)^\perp$, because for $\mathscr{E}'$ in $\mathbb{E}(X)$ we have $\operatorname{Hom}(\mathscr{E}', f_*(\mathscr{Y})) \cong \operatorname{Hom}(\mathscr{E}'|_V, \mathscr{Y}) = 0$. This proves that $f_*(\mathscr{T})$ belongs to the subcategory $\mathcal{L}$, and completes the proof of the theorem. □

Remark 3.17. For affine schemes the quotient $Q : \mathbb{K}(\text{Flat } A) \longrightarrow \mathbb{K}_m(\text{Proj } A)$ has a left adjoint, so the localization sequence of the theorem is a recollement; see Remark 3.5(*iii*). This is false for arbitrary schemes, and $X = \mathbb{P}^1_k$ gives a counterexample; see Remark A.15. The point is that when Q has a left adjoint products in $\mathfrak{Qco}(X)$ must be exact, and this is known to fail for the projective line.

3.2 Enough Flat Quasi-coherent Sheaves

Over a quasi-projective variety any quasi-coherent sheaf can be written as a quotient of a locally free sheaf; see [TT90, Lemma 2.1.3]. In particular, every quasi-coherent sheaf is a quotient of a flat quasi-coherent sheaf. We prove in this section that this weaker condition holds for all schemes (recall that, by our standing hypothesis, schemes are quasi-compact and separated). This fact will become very important in Chapter 5.

The key observation is contained in Proposition 3.19. First, we give a technical lemma that is well-known, but for which we could not find a convenient reference.

Setup. In this section X denotes a scheme, and sheaves are defined over X by default.

Lemma 3.18. *Let $\mathcal{T}$ be a triangulated category, $\mathcal{S}$ a thick triangulated subcategory, and suppose that we have a commutative diagram in $\mathcal{T}$ with triangles for rows*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, "h"'] & \Sigma A \arrow[d] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & \Sigma A' \end{tikzcd} \tag{3.23}$$

If any two of f, g, h have mapping cone in $\mathcal{S}$, then so does the third.

Proof. Assume without loss of generality that f, g have mapping cone in $\mathcal{S}$, and therefore determine isomorphisms in the quotient $\mathcal{T}/\mathcal{S}$. Applying [Nee01b, Proposition 1.1.20] to the image of (3.23) in $\mathcal{T}/\mathcal{S}$ we infer that h is an isomorphism in $\mathcal{T}/\mathcal{S}$. Using the fact that $\mathcal{S}$ is thick and [Nee01b, Proposition 2.1.35] we conclude that h has mapping cone in $\mathcal{S}$. □

Flat resolutions are not unique in the homotopy category, and in this sense they are inferior to projective and injective resolutions. One solution is to work with a more rigid kind of resolution, known as a *proper* resolution. Suppose that we have an exact sequence of modules over a ring

$$S : \cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow P^0 \longrightarrow M \longrightarrow 0 \tag{3.24}$$

If this is a projective resolution of M then a morphism from a projective module to a kernel of the complex S (for example M itself, or its syzygy $Ker(P^0 \longrightarrow M)$) must factor through the relevant projective object (for example P^0 or P^{-1}). This property allows one to prove that the projective resolution is unique up to homotopy equivalence.

If a flat resolution has this property with respect to morphisms from flat modules, it is called a *proper flat resolution*. This is a flat resolution (3.24) with the additional property that $P^0 \longrightarrow M$ and $P^{-1} \longrightarrow Ker(P^0 \longrightarrow M)$ and so on are all *flat precovers* in the sense of Definition 2.32. We review the relevant literature in Section 2.3, but for our purposes in this section only two facts are really necessary: proper (left) flat resolutions exist, and can be encoded in triangles; see Definition 2.33 and Corollary 2.36. In the next result we use proper flat resolutions of modules to prove a useful fact about quasi-coherent sheaves.

Proposition 3.19. *Every quasi-coherent sheaf $\mathscr{F}$ is isomorphic in $\mathbb{D}(\mathfrak{Qco}X)$ to a bounded above complex of flat quasi-coherent sheaves.*

Proof. This is implicit in the proof of [AJL97, Proposition 1.1] but we give another proof using flat precovers. Let $\mathfrak{U} = \{U_0, \dots, U_d\}$ be an affine open cover of X, and consider the Čech resolution

$$0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{C}^1(\mathfrak{U}, \mathscr{F}) \longrightarrow \cdots \longrightarrow \mathscr{C}^d(\mathfrak{U}, \mathscr{F}) \longrightarrow 0$$

This defines a quasi-isomorphism $\mathscr{F} \longrightarrow \mathscr{C}(\mathfrak{U}, \mathscr{F})$ so it is enough to prove that the complex $\mathscr{C}(\mathfrak{U}, \mathscr{F})$ is isomorphic, in $\mathbb{D}(\mathfrak{Qco}X)$, to a bounded above complex of flat quasi-coherent sheaves. This is true, because locally $\mathscr{F}$ has proper flat resolutions.

In detail: given a finite sequence $\alpha : i_0 < \cdots < i_p$ of indices in the set $\{0, \dots, d\}$ we write $U_\alpha = U_{i_0} \cap \cdots \cap U_{i_p}$ for the corresponding intersection; this is an affine scheme, so we can find a proper left flat resolution of $\mathscr{F}|_{U_\alpha}$ in the sense of Definition 2.33

$$\mathscr{S}_\alpha : \cdots \longrightarrow \mathscr{P}_\alpha^{-2} \longrightarrow \mathscr{P}_\alpha^{-1} \longrightarrow \mathscr{P}_\alpha^0 \longrightarrow \mathscr{F}|_{U_\alpha} \longrightarrow 0 \tag{3.25}$$

To be clear, we choose an isomorphism $U_\alpha \cong Spec(A)$ and use the equivalence $\mathfrak{Qco}(U_\alpha) \cong$ $\mathbf{Mod}A$ to pair $\mathscr{F}|_{U_\alpha}$ with an A-module F_α. Then we take a proper left flat resolution

$$S_\alpha : \cdots \longrightarrow P_\alpha^{-2} \longrightarrow P_\alpha^{-1} \longrightarrow P_\alpha^0 \longrightarrow F_\alpha \longrightarrow 0 \tag{3.26}$$

which is an exact sequence with each P_α^i flat, such the canonical morphism of complexes $P_\alpha \longrightarrow F_\alpha$ fits into a triangle $P_\alpha \longrightarrow F_\alpha \longrightarrow S_\alpha \longrightarrow \Sigma P_\alpha$ in $\mathbb{K}(A)$ with S_α in $\mathbb{K}(\mathrm{Flat}\, A)^\perp$. This resolution exists because flat precovers exist; see Remark 2.34 and Corollary 2.36.

Passing back to $\mathfrak{Qco}(U_\alpha)$ we have an exact sequence (3.25) of quasi-coherent sheaves that fits into a triangle in $\mathbb{K}(\mathfrak{Qco}U_\alpha)$ with $\mathscr{P}_\alpha$ in $\mathbb{K}(\mathrm{Flat}\, U_\alpha)$ and $\mathscr{S}_\alpha$ in $\mathbb{K}(\mathrm{Flat}\, U_\alpha)^\perp$

$$\mathscr{P}_\alpha \longrightarrow \mathscr{F}|_{U_\alpha} \longrightarrow \mathscr{S}_\alpha \longrightarrow \Sigma \mathscr{P}_\alpha \tag{3.27}$$

Moreover, the complex $\mathscr{S}_\alpha$ is acyclic. The inclusion $f : U_\alpha \longrightarrow X$ is flat and affine, so the direct image sends (3.27) to a triangle in $\mathbb{K}(\mathfrak{Qco}X)$

$$f_*(\mathscr{P}_\alpha) \longrightarrow f_*(\mathscr{F}|_{U_\alpha}) \longrightarrow f_*(\mathscr{S}_\alpha) \longrightarrow \Sigma f_*(\mathscr{P}_\alpha)$$

with $f_*(\mathscr{P}_\alpha) \in \mathbb{K}(\mathrm{Flat}\, X)$ and $f_*(\mathscr{S}_\alpha) \in \mathbb{K}_{\mathrm{ac}}(\mathfrak{Qco}X) \cap \mathbb{K}(\mathrm{Flat}\, X)^\perp$. To see that $f_*(\mathscr{S}_\alpha)$ belongs to the orthogonal $\mathbb{K}(\mathrm{Flat}\, X)^\perp$ we use the adjunction between direct image and restriction. Taking the coproduct over all sequences $\alpha : i_0 < \cdots < i_p$ of length p, we have a triangle in $\mathbb{K}(\mathfrak{Qco}X)$

$$\mathscr{P}_p \longrightarrow \mathscr{C}^p(\mathfrak{U}, \mathscr{F}) \longrightarrow \mathscr{S}_p \longrightarrow \Sigma \mathscr{P}_p$$

with $\mathscr{P}_p \in \mathbb{K}(\operatorname{Flat} X)$ and $\mathscr{S}_p \in \mathbb{K}_{\mathrm{ac}}(\mathfrak{Qco} X) \cap \mathbb{K}(\operatorname{Flat} X)^{\perp}$. In particular $\mathscr{P}_p \longrightarrow \mathscr{C}^p(\mathfrak{U}, \mathscr{F})$ is a quasi-isomorphism. This shows that the individual Čech sheaves are quasi-isomorphic to bounded above complexes of flat quasi-coherent sheaves, and it remains to argue that these resolutions can be combined to give a resolution for the complex $\mathscr{C}(\mathfrak{U}, \mathscr{F})$.

Any bounded complex can be built, in a finite number of triangles, from the objects occurring in the complex. Using *proper* flat resolutions provides enough ridigity for us to assemble the resolutions of the Čech sheaves into a resolution of $\mathscr{C}(\mathfrak{U}, \mathscr{F})$. The complex $\mathscr{C}(\mathfrak{U}, \mathscr{F})$ is the mapping cone of the following morphism of complexes (each arranged in the correct degree)

$$\begin{tikzcd}
\Sigma^{-1}\mathscr{C}^0(\mathfrak{U},\mathscr{F}) : \arrow[d] & \cdots \arrow[r] & 0 \arrow[r] & \mathscr{C}^0(\mathfrak{U},\mathscr{F}) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \cdots \\
{}_b\mathscr{C}(\mathfrak{U},\mathscr{F})_{\geq 1} : & \cdots \arrow[r] & 0 \arrow[r] & \mathscr{C}^1(\mathfrak{U},\mathscr{F}) \arrow[r] & \mathscr{C}^2(\mathfrak{U},\mathscr{F}) \arrow[r] & \cdots
\end{tikzcd}$$

More generally, we can adjoin the kth Čech sheaf to the brutal truncation ${}_b\mathscr{C}(\mathfrak{U}, \mathscr{F})_{\geq k+1}$ to form ${}_b\mathscr{C}(\mathfrak{U}, \mathscr{F})_{\geq k}$ (our notation for truncations is given in Section 2.1). This is reflected by a triangle in $\mathbb{K}(\mathfrak{Qco} X)$

$$\Sigma^{-k-1}\mathscr{C}^k(\mathfrak{U},\mathscr{F}) \longrightarrow {}_b\mathscr{C}(\mathfrak{U},\mathscr{F})_{\geq k+1} \longrightarrow {}_b\mathscr{C}(\mathfrak{U},\mathscr{F})_{\geq k} \longrightarrow \Sigma^{-k}\mathscr{C}^k(\mathfrak{U},\mathscr{F})$$

Suppose that for some integer $0 < k \leq d$ we have defined a bounded above complex of flat quasi-coherent sheaves $\mathscr{A}_k$ and a quasi-isomorphism $\mathscr{A}_k \longrightarrow {}_b\mathscr{C}(\mathfrak{U}, \mathscr{F})_{\geq k}$ with mapping cone $\mathscr{B}_k$ in $\mathbb{K}(\operatorname{Flat} X)^{\perp}$. This has been done for $k = d$, where ${}_b\mathscr{C}(\mathfrak{U}, \mathscr{F})_{\geq d} = \Sigma^{-d}\mathscr{C}^d(\mathfrak{U}, \mathscr{F})$ and we can take $\mathscr{A}_d = \Sigma^{-d}\mathscr{P}_d$ and $\mathscr{B}_d = \Sigma^{-d}\mathscr{S}_d$ as defined above. Returning to the case of general $0 < k \leq d$ we have a diagram with triangles for rows

$$\begin{tikzcd}
\Sigma^{-k}\mathscr{P}_{k-1} \arrow[r] \arrow[d, dotted] \arrow[dr, phantom, "\mathrm{(I)}"] & \Sigma^{-k}\mathscr{C}^{k-1}(\mathfrak{U},\mathscr{F}) \arrow[r] \arrow[d] & \Sigma^{-k}\mathscr{S}_{k-1} \arrow[r] \arrow[d, dotted] & \Sigma^{-k+1}\mathscr{P}_{k-1} \arrow[d, dotted] \\
\mathscr{A}_k \arrow[r] & {}_b\mathscr{C}(\mathfrak{U},\mathscr{F})_{\geq k} \arrow[r] & \mathscr{B}_k \arrow[r] & \Sigma\mathscr{A}_k
\end{tikzcd} \tag{3.28}$$

The following composite vanishes

$$\Sigma^{-k}\mathscr{P}_{k-1} \longrightarrow \Sigma^{-k}\mathscr{C}^{k-1}(\mathfrak{U},\mathscr{F}) \longrightarrow {}_b\mathscr{C}(\mathfrak{U},\mathscr{F})_{\geq k} \longrightarrow \mathscr{B}_k$$

because $\mathscr{P}_{k-1} \in \mathbb{K}(\operatorname{Flat} X)$ and $\mathscr{B}_k \in \mathbb{K}(\operatorname{Flat} X)^{\perp}$ (this is the point where we use the properness of our resolutions). We deduce unique vertical morphisms making (3.28) into a morphism of triangles. Now extend the commutative square in (3.28) marked (I) to a morphism of triangles in the vertical direction

$$\begin{tikzcd}
\Sigma^{-k}\mathscr{P}_{k-1} \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{(I)}"] & \mathscr{A}_k \arrow[r] \arrow[d, "g"'] & \mathscr{A}_{k-1} \arrow[r] \arrow[d, "h"'] & \Sigma^{-k+1}\mathscr{P}_{k-1} \arrow[d] \\
\Sigma^{-k}\mathscr{C}^{k-1}(\mathfrak{U},\mathscr{F}) \arrow[r] & {}_b\mathscr{C}(\mathfrak{U},\mathscr{F})_{\geq k} \arrow[r] & {}_b\mathscr{C}(\mathfrak{U},\mathscr{F})_{\geq k-1} \arrow[r] & \Sigma^{-k+1}\mathscr{C}^{k-1}(\mathfrak{U},\mathscr{F})
\end{tikzcd}$$

As f, g both have mapping cone in $\mathbb{K}_{\mathrm{ac}}(\mathfrak{Qco} X) \cap \mathbb{K}(\operatorname{Flat} X)^{\perp}$ it follows from Lemma 3.18 that h has mapping cone $\mathscr{B}_{k-1}$ in this subcategory, which completes the inductive step. Taking $k = 1$ we deduce a quasi-isomorphism $\mathscr{A}_0 \longrightarrow \mathscr{C}(\mathfrak{U}, \mathscr{F})$ with $\mathscr{A}_0$ a bounded above complex of flat quasi-coherent sheaves, as required. $\square$

In the next lemma, let $\mathcal{A}$ be an abelian category and $\mathcal{P} \subseteq \mathcal{A}$ a class containing the zero objects and closed under isomorphism, with the property that for any short exact sequence $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ in $\mathcal{A}$ with $M, N \in \mathcal{P}$ we have also $L \in \mathcal{P}$. For example, this applies when $\mathcal{A}$ is the category of quasi-coherent sheaves and $\mathcal{P}$ the class of flat quasi-coherent sheaves (or vector bundles over a noetherian scheme).

Lemma 3.20. *Let $\mathcal{A}$ be an abelian category and $\mathcal{P} \subseteq \mathcal{A}$ a class as above. If $M \in \mathcal{A}$ is isomorphic in $\mathbb{D}(\mathcal{A})$ to a bounded above complex in $\mathcal{P}$, there exists an exact sequence*

$$\cdots \longrightarrow F^{-2} \longrightarrow F^{-1} \longrightarrow F^{0} \longrightarrow M \longrightarrow 0 \tag{3.29}$$

with every F^i an object of $\mathcal{P}$.

Proof. Let P be a bounded above complex in $\mathcal{P}$, isomorphic in $\mathbb{D}(\mathcal{A})$ to the object M. Assume that M is nonzero, and let d be the largest integer with $P^d \neq 0$. If $d = 0$ then we are done. If $d > 0$ then we have a series of short exact sequences

$$\begin{gathered}
0 \longrightarrow K^{d-1} \longrightarrow P^{d-1} \longrightarrow P^{d} \longrightarrow 0 \\
0 \longrightarrow K^{d-2} \longrightarrow P^{d-2} \longrightarrow K^{d-1} \longrightarrow 0 \\
\vdots \\
0 \longrightarrow K^{1} \longrightarrow P^{1} \longrightarrow K^{2} \longrightarrow 0 \\
0 \longrightarrow K^{0} \longrightarrow P^{0} \longrightarrow K^{1} \longrightarrow 0
\end{gathered}$$

where $K^i = Ker(P^i \longrightarrow P^{i+1})$. We deduce that K^0 belongs to $\mathcal{P}$, so

$$\cdots \longrightarrow P^{-2} \longrightarrow P^{-1} \longrightarrow K^{0} \longrightarrow M \longrightarrow 0$$

is a resolution of the desired form. $\square$

Corollary 3.21. *Every quasi-coherent sheaf $\mathscr{F}$ admits an epimorphism $\mathscr{P} \longrightarrow \mathscr{F}$ with $\mathscr{P}$ a flat quasi-coherent sheaf.*

Proof. By Proposition 3.19 the sheaf $\mathscr{F}$ is isomorphic, in $\mathbb{D}(\mathfrak{Qco} X)$, to a bounded above complex of flat quasi-coherent sheaves. If we take $\mathcal{P}$ to be the class of flat quasi-coherent sheaves in $\mathcal{A} = \mathfrak{Qco}(X)$, then Lemma 3.20 provides an epimorphism $\mathscr{P} \longrightarrow \mathscr{F}$ with $\mathscr{P}$ a flat quasi-coherent sheaf. $\square$

Corollary 3.22. *Any complex $\mathscr{F}$ of quasi-coherent sheaves admits a quasi-isomorphism $\mathscr{P} \longrightarrow \mathscr{F}$ with $\mathscr{P}$ a $\mathbb{K}$-flat complex of flat quasi-coherent sheaves which is the homotopy colimit in $\mathbb{K}(\mathrm{Flat}\, X)$ of a sequence*

$$\mathscr{P}_0 \longrightarrow \mathscr{P}_1 \longrightarrow \mathscr{P}_2 \longrightarrow \mathscr{P}_3 \longrightarrow \cdots \tag{3.30}$$

of bounded above complexes $\mathscr{P}_i$ of flat quasi-coherent sheaves.

Proof. By Corollary 3.21 the class $\mathcal{P}$ of flat quasi-coherent sheaves satisfies the hypotheses needed for Lemma 2.22, which then constructs a quasi-isomorphism $\mathscr{P} \longrightarrow \mathscr{F}$ with $\mathscr{P}$ a complex of flat quasi-coherent sheaves. Moreover, $\mathscr{P}$ is the homotopy colimit in $\mathbb{K}(\mathfrak{Qco}X)$ of a sequence (3.30) of bounded above complexes of flat quasi-coherent sheaves. Because each $\mathscr{P}_i$ is $\mathbb{K}$-flat the homotopy colimit $\mathscr{P}$ is $\mathbb{K}$-flat, completing the proof. □

Remark 3.23. Enochs and Estrada have shown that flat precovers exist in the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves on X [EE05a, Corollary 4.2]. From Corollary 3.21 we learn that flat precovers in $\mathfrak{Qco}(X)$ are always epimorphisms, which answers an implicit question of Enochs and Estrada in [EE05a, §5]. If we knew *a priori* that flat precovers were epimorphisms in $\mathfrak{Qco}(X)$, then Corollary 3.21 would be unnecessary.

The reader may safely skip the next pair of results, which will not be used elsewhere. We include them to clarify a small point in the definition of the category $\mathbb{E}(X)$.

Lemma 3.24. *For a quasi-coherent sheaf $\mathscr{F}$ the following are equivalent:*

(i) $\mathscr{F}$ is flat.

(ii) $\mathscr{T}or_i(\mathscr{F}, \mathscr{G}) = 0$ for every quasi-coherent sheaf $\mathscr{G}$ and $i > 0$.

(iii) For every exact sequence of quasi-coherent sheaves

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \mathscr{C} \longrightarrow 0$$

the following sequence is also exact

$$0 \longrightarrow \mathscr{F} \otimes \mathscr{A} \longrightarrow \mathscr{F} \otimes \mathscr{B} \longrightarrow \mathscr{F} \otimes \mathscr{C} \longrightarrow 0$$

Proof. $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$ are obvious. $(ii) \Rightarrow (i)$ We prove that $\mathscr{F}$ is flat by showing that $\mathscr{F}_x$ is a flat $\mathcal{O}_{X,x}$-module for every $x \in X$. Let an $\mathcal{O}_{X,x}$-module G be given. We can find a quasi-coherent sheaf $\mathscr{G}$ on X with stalk G at x (take an affine open neighborhood $U \cong Spec(A)$ of x with inclusion $f : U \longrightarrow X$ and let $\mathscr{G}$ be $f_*(G^\sim)$). Then for $i > 0$ we have $0 = \mathscr{T}or_i(\mathscr{F}, \mathscr{G})_x \cong \mathrm{Tor}_i(\mathscr{F}_x, G)$. Since G was arbitrary this proves that $\mathscr{F}_x$ is flat. $(iii) \Rightarrow (i)$ Given $x \in X$ find an affine open neighborhood $U \cong Spec(A)$ of x, and let $f : U \longrightarrow X$ denote the inclusion. Let $\mathscr{A} \longrightarrow \mathscr{B}$ be a monomorphism of quasi-coherent sheaves on U. Then $f_*\mathscr{A} \longrightarrow f_*\mathscr{B}$ is a monomorphism, whence $\mathscr{F} \otimes f_*(\mathscr{A}) \longrightarrow \mathscr{F} \otimes f_*(\mathscr{B})$ is a monomorphism, and restricting again to U we find that $\mathscr{F}|_U \otimes \mathscr{A} \longrightarrow \mathscr{F}|_U \otimes \mathscr{B}$ is a monomorphism. If M is an A-module with $\mathscr{F}|_U \cong M^\sim$ then we deduce that M is flat. If $\mathfrak{p}$ is the prime ideal corresponding to x, then $\mathscr{F}_x \cong M_\mathfrak{p}$ is a flat module over $\mathcal{O}_{X,x} \cong A_\mathfrak{p}$. Since x was arbitrary, this proves that $\mathscr{F}$ is flat. □

Lemma 3.25. *A complex $\mathscr{E}$ of flat quasi-coherent sheaves belongs to $\mathbb{E}(X)$ if and only if $\mathscr{F} \otimes \mathscr{E}$ is acyclic for every* quasi-coherent *sheaf $\mathscr{F}$.*

Proof. If $\mathscr{E}$ belongs to $\mathbb{E}(X)$ then it has the stated property. For the converse, assume that $\mathscr{F} \otimes \mathscr{E}$ is acyclic for every quasi-coherent sheaf $\mathscr{F}$. From the proof of $(i) \Rightarrow (ii)$ in Proposition 3.4 we learn that $\mathscr{T}or_i(\mathscr{F}, Ker(\partial_{\mathscr{E}}^n))$ is zero for every quasi-coherent sheaf $\mathscr{F}$ and $i > 0$, so Lemma 3.24 implies that $Ker(\partial_{\mathscr{E}}^n)$ is flat. Hence $\mathscr{E}$ is an acyclic complex of flat quasi-coherent sheaves with flat kernels, and therefore an object of $\mathbb{E}(X)$. $\square$

Chapter 4

Compact Generation of $\mathbb{K}_m(\operatorname{Proj} X)$

This chapter contains our proof that $\mathbb{K}_m(\operatorname{Proj} X)$ is compactly generated for a noetherian scheme X. In outline, here is the proof: an affine open cover of X determines a cocovering of $\mathbb{K}_m(\operatorname{Proj} X)$ by Bousfield subcategories, and the application of a theorem of Rouquier [Rou03, Theorem 5.15] to this cocover reduces the theorem to a statement about complexes of projective modules over a ring.

For a noetherian ring A we know that the homotopy category $\mathbb{K}(\operatorname{Proj} A)$ of projective A-modules is compactly generated, by the work of Jørgensen and Neeman (Theorem 2.30). To apply Rouquier's theorem to our cocover, we need something more: we need to prove that for $f_1, \ldots, f_r \in A$ the intersection over $1 \le i \le r$ of the kernels of the localizations

$$- \otimes_A A_{f_i} : \mathbb{K}(\operatorname{Proj} A) \longrightarrow \mathbb{K}(\operatorname{Proj} A_{f_i})$$

is compactly generated in $\mathbb{K}(\operatorname{Proj} A)$. An element of this intersection of kernels is a complex of projective A-modules that becomes contractible after localizing at each f_i. We introduce the following notation for the triangulated subcategory of such complexes

$$\mathcal{V}(f_1, \ldots, f_r) = \{P \in \mathbb{K}(\operatorname{Proj} A) \,|\, P_{f_i} \text{ is zero in } \mathbb{K}(\operatorname{Proj} A_{f_i}) \text{ for } 1 \le i \le r\}$$

In Section 2.3 we studied how to produce compact objects in $\mathbb{K}(\operatorname{Proj} A)$: given a finitely generated A-module M, the complex of projective A-modules P_M that "represents" M is compact in $\mathbb{K}(\operatorname{Proj} A)$. This representing complex of projectives is called the *proper right projective resolution*; we recall the details in Definition 4.3 below. With this in mind, here is our recipe for producing compact objects of the category $\mathcal{V}(f_1, \ldots, f_r)$:

(a) Let M be a finitely generated A-module with proper right projective resolution P_M.

(b) For $f \in A$ the Koszul complex $K(f) = 0 \longrightarrow A \xrightarrow{f} A \longrightarrow 0$ becomes contractible after localizing at f, and given integers $B_1, \ldots, B_r > 0$ the tensor product

$$K(f_1^{B_1}) \otimes \cdots \otimes K(f_r^{B_r}) \otimes P_M \tag{4.1}$$

belongs to $\mathcal{V}(f_1, \ldots, f_r)$. We prove in Lemma 4.4 that it is compact.

In Proposition 4.5 we prove that the complexes of the form (4.1) form a compact generating set for $\mathcal{V}(f_1, \dots, f_r)$ consisting of objects compact in the larger category $\mathbb{K}(\operatorname{Proj} A)$. This removes the only obstacle to applying Rouquier's theorem, so we proceed in Section 4.1 with the proof that $\mathbb{K}_m(\operatorname{Proj} X)$ is compactly generated.

Setup. In this section A is a noetherian ring, and modules are defined over A by default. Throughout we fix a family of elements $f_1, \dots, f_r \in A$.

We begin with Koszul complexes, which are complexes that become contractible after localization for the most trivial reason possible.

Definition 4.1. Given an element $f \in A$ the corresponding *Koszul complex* $K(f)$ is the following complex concentrated in degrees $-1, 0$

$$\cdots \longrightarrow 0 \longrightarrow A \xrightarrow{f} A \longrightarrow 0 \longrightarrow \cdots \tag{4.2}$$

For a family $h_1, \dots, h_n \in A$ we introduce the notation

$$K(h_1, \dots, h_n) := K(h_1) \otimes \cdots \otimes K(h_n)$$

Clearly $K(f)$ becomes contractible after localizing at f. For a complex of modules X the tensor product $K(f) \otimes X$ is isomorphic to the mapping cone of $f = f \cdot 1 : X \longrightarrow X$, so there is a triangle $X \xrightarrow{f} X \longrightarrow K(f) \otimes X \longrightarrow \Sigma X$ in $\mathbb{K}(A)$.

The next lemma tells us that Koszul complexes arise naturally.

Lemma 4.2. *Let $\varphi : M \longrightarrow X$ be a morphism of complexes of modules, with M bounded and M^i finitely generated for every $i \in \mathbb{Z}$. Then*

(i) Given $f \in A$ with X_f zero in $\mathbb{K}(A_f)$ there exists an integer $B > 0$ such that $f^B \cdot \varphi$ is null-homotopic.

(ii) If X_{f_i} is zero in $\mathbb{K}(A_{f_i})$ for every $1 \le i \le r$ then there are integers $B_1, \dots, B_r > 0$ such that the morphism φ can be factored in $\mathbb{K}(A)$ as

$$M \longrightarrow K(f_1^{B_1}, \dots, f_r^{B_r}) \otimes M \longrightarrow X$$

Proof. (*i*) Let $\iota : X \longrightarrow X_f$ be the canonical morphism of complexes of A-modules, and choose a contracting homotopy Λ of X_f as a complex of A_f-modules. We have a diagram

$$\begin{tikzcd}
\cdots \arrow[r] & M^{i-1} \arrow[r] \arrow[d, "\varphi^{i-1}"] & M^i \arrow[r] \arrow[d, "\varphi^i"] & M^{i+1} \arrow[r] \arrow[d, "\varphi^{i+1}"] & \cdots \\
\cdots \arrow[r] & X^{i-1} \arrow[r] \arrow[d] & X^i \arrow[r] \arrow[d] & X^{i+1} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r, shift left] & X_f^{i-1} \arrow[r, shift left] \arrow[l, shift left] & X_f^i \arrow[r, shift left] \arrow[l, shift left, "\Lambda^i"] & X_f^{i+1} \arrow[r, shift left] \arrow[l, shift left, "\Lambda^{i+1}"] & \cdots \arrow[l, shift left]
\end{tikzcd}$$

For $i \in \mathbb{Z}$ there is an isomorphism $\mathrm{Hom}_A(M^i, X_f^{i-1}) \xrightarrow{\sim} Hom_A(M^i, X^{i-1})_f$ which must identify $\Lambda^i \iota^i \varphi^i$ with a fraction λ^i / f^{k_i} for some morphism $\lambda^i : M^i \longrightarrow X^{i-1}$ and $k_i > 0$. Because M is bounded, we can take the k_i to be equal to some fixed integer $k > 0$, so that

$$\iota^{i-1}\lambda^i = f^k \Lambda^i \iota^i \varphi^i \quad \text{for all } i \in \mathbb{Z}$$

Since Λ is a contracting homotopy for X_f, there is an equality for every $i \in \mathbb{Z}$

$$\partial_{X_f}^{i-1}\Lambda^i + \Lambda^{i+1}\partial_{X_f}^i = 1$$

Composing with $\iota^i\varphi^i$ and multiplying by f^k yields

$$\partial_{X_f}^{i-1}\iota^{i-1}\lambda^i + \iota^i\lambda^{i+1}\partial_M^i = f^k\iota^i\varphi^i$$

This equality in $\mathrm{Hom}_A(M^i, X_f^i)$ determines an equality in $\mathrm{Hom}_A(M^i, X^i)_f$ of fractions $(\partial_X^{i-1}\lambda^i + \lambda^{i+1}\partial_M^i)/1 = (f^k\varphi^i)/1$. We can make this into an equality in $\mathrm{Hom}_A(M^i, X^i)$ by multiplying the numerators by a sufficiently high power of f, and since M is bounded we can make a fixed integer $N > 0$ work for every $i \in \mathbb{Z}$. That is,

$$f^N(\partial_X^{i-1}\lambda^i + \lambda^{i+1}\partial_M^i) = f^{N+k}\varphi^i \quad \text{for all } i \in \mathbb{Z}$$

If we set $B = N + k$ then $\mu^i = f^N\lambda^i$ gives a homotopy of $f^B \cdot \varphi$ with zero, as required.

(ii) The proof is by induction on $r \geq 1$. For $r = 1$ we apply (i) to find an integer $B > 0$ and a null-homotopy of the morphsim $f_1^B \cdot \varphi$. From the triangle

$$M \xrightarrow{f_1^B} M \longrightarrow K(f_1^B) \otimes M \longrightarrow \Sigma M$$

we obtain the required factorization of $\varphi : M \longrightarrow X$ through $M \longrightarrow K(f_1^B) \otimes M$. Suppose that $r > 1$ is given and assume the lemma for all smaller values of r. In particular, we can factor φ as a composite in $\mathbb{K}(A)$ of the form

$$M \longrightarrow K(f_1^{B_1}, \ldots, f_{r-1}^{B_{r-1}}) \otimes M \longrightarrow X \tag{4.3}$$

for some integers $B_1, \ldots, B_{r-1} > 0$. The complex $G = K(f_1^{B_1}, \ldots, f_{r-1}^{B_{r-1}}) \otimes M$ is bounded and has finitely generated terms, so we can apply the case $r = 1$ to the morphism $G \longrightarrow X$ coming from (4.3) to obtain a factorization of φ as the following composite in $\mathbb{K}(A)$

$$M \longrightarrow G \longrightarrow K(f_r^{B_r}) \otimes G \longrightarrow X$$

Since $K(f_r^{B_r}) \otimes G \cong K(f_1^{B_1}, \ldots, f_r^{B_r}) \otimes M$ this completes the proof. $\square$

For a complex P of projective modules, $K(f_1^{B_1}, \ldots, f_r^{B_r}) \otimes P$ belongs to $\mathcal{V}(f_1, \ldots, f_r)$ for any integers $B_1, \ldots, B_r > 0$. To produce compact objects we apply this construction when P is the proper right projective resolution of a finitely generated module. Let us tell the reader what these proper resolutions are.

Proper resolutions belong to the subject of relative homological algebra, and we give an exposition of the relevant theory in Section 2.3. Our approach is slightly different to the literature, because we need to talk about proper resolutions of complexes (for modules, our terminology agrees with the standard definitions). The following is Definition 2.37 in the case $\mathcal{X} = \mathrm{Proj}(A)$, the class of projective A-modules.

Definition 4.3. Let G be a complex of modules. A *proper right projective resolution* of G is a morphism of complexes $G \longrightarrow P_G$ fitting into a triangle in $\mathbb{K}(A)$

$$S \longrightarrow G \longrightarrow P_G \longrightarrow \Sigma S$$

with P_G in $\mathbb{K}(\operatorname{Proj} A)$ and S in the orthogonal ${}^{\perp}\mathbb{K}(\operatorname{Proj} A)$. Given $Q \in \mathbb{K}(\operatorname{Proj} A)$ we can apply $\operatorname{Hom}_{\mathbb{K}(A)}(-, Q)$ to this triangle to obtain a natural isomorphism

$$\operatorname{Hom}_{\mathbb{K}(\operatorname{Proj} A)}(P_G, Q) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}(A)}(G, Q) \tag{4.4}$$

Hence P_G *represents* G amongst the complexes of projective modules.

Every finitely generated module has a proper right projective resolution by Remark 2.26. It follows that every bounded complex of finitely generated modules has a proper right projective resolution, but we are only interested in resolutions of very special bounded complexes: those of the form $K(f_1^{B_1}, \ldots, f_r^{B_r}) \otimes M$ for some finitely generated module M, and for these complexes we can describe the resolution explicitly.

Lemma 4.4. *Let M be a finitely generated module with proper right projective resolution P_M. Given $B_1, \ldots, B_r > 0$ the complex of projective modules*

$$K(f_1^{B_1}, \ldots, f_r^{B_r}) \otimes P_M \tag{4.5}$$

is a proper right projective resolution of $K(f_1^{B_1}, \ldots, f_r^{B_r}) \otimes M$. It follows that the complex (4.5) is a compact object in $\mathbb{K}(\operatorname{Proj} A)$.

Proof. By definition we have a triangle $S \longrightarrow M \longrightarrow P_M \longrightarrow \Sigma S$ in $\mathbb{K}(A)$ where S belongs to the orthogonal ${}^{\perp}\mathbb{K}(\operatorname{Proj} A)$. Tensoring with $K = K(f_1^{B_1}, \ldots, f_r^{B_r})$ we have a triangle

$$K \otimes S \longrightarrow K \otimes M \longrightarrow K \otimes P_M \longrightarrow \Sigma K \otimes S$$

To prove that $K \otimes P_M$ is a proper right projective resolution of $K \otimes M$, we need only show that $K \otimes S$ is left orthogonal to every complex Q of projective modules. But

$$\operatorname{Hom}_{\mathbb{K}(A)}(K \otimes S, Q) \cong \operatorname{Hom}_{\mathbb{K}(A)}(S, \operatorname{Hom}_A(K, Q)) = 0$$

as $\operatorname{Hom}_A(K, Q)$ is a complex of projective modules, so $K \otimes S$ belongs to ${}^{\perp}\mathbb{K}(\operatorname{Proj} A)$ and $K \otimes P_M$ is a proper right projective resolution of $K \otimes M$. By definition, or more precisely (4.4), we have a natural isomorphism

$$\operatorname{Hom}_{\mathbb{K}(\operatorname{Proj} A)}(K \otimes P_M, -) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}(A)}(K \otimes M, -) \tag{4.6}$$

Because $K \otimes M$ is a bounded complex of finitely generated modules it is compact in $\mathbb{K}(A)$, and from (4.6) we conclude that $K \otimes P_M$ is compact in $\mathbb{K}(\operatorname{Proj} A)$. $\square$

Finally, we construct a compact generating set for $\mathcal{V}(f_1, \ldots, f_r)$. Recall from Chapter 2 that a localizing subcategory $\mathcal{S}$ of a triangulated category $\mathcal{T}$ is *compactly generated in* $\mathcal{T}$ if it has a compact generating set consisting of objects compact in $\mathcal{T}$.

Proposition 4.5. *The subcategory $\mathcal{V}(f_1,\dots,f_r)$ is compactly generated in $\mathbb{K}(\operatorname{Proj} A)$ with compact generating set*

$$\mathcal{R} = \big\{\Sigma^j K(f_1^{B_1},\dots,f_r^{B_r}) \otimes P_M \,|\, M \text{ is a finitely generated module, } B_1,\dots,B_r > 0 \text{ and } j \in \mathbb{Z}\big\}$$

where P_M denotes a proper right projective resolution of M.

Proof. Up to isomorphism there is a set of finitely generated modules. Pick one module M from each isomorphism class and let P_M be a proper right projective resolution. Tensoring with Koszul complexes and shifting gives the set $\mathcal{R}$, which by Lemma 4.4 is a set of objects of $\mathcal{V}(f_1,\dots,f_r)$ compact in $\mathbb{K}(\operatorname{Proj} A)$. We claim that this set generates $\mathcal{V}(f_1,\dots,f_r)$.

Given a nonzero object $X \in \mathcal{V}(f_1,\dots,f_r)$ we have to find a finitely generated module M, integers $B_1,\dots,B_r > 0$ and $j \in \mathbb{Z}$, and a nonzero morphism in $\mathbb{K}(\operatorname{Proj} A)$ (from now on, *nonzero* means nonzero in the homotopy category)

$$\Sigma^j K(f_1^{B_1},\dots,f_r^{B_r}) \otimes P_M \longrightarrow X \tag{4.7}$$

By Theorem 2.30 the proper resolutions of finitely generated modules compactly generate $\mathbb{K}(\operatorname{Proj} A)$, and since X is nonzero in $\mathbb{K}(\operatorname{Proj} A)$ there must be a finitely generated module M with proper right projective resolution P_M and a nonzero morphism

$$\Sigma^j P_M \longrightarrow X \tag{4.8}$$

After shifting we may assume that $j = 0$. By assumption the localization X_{f_i} is zero in $\mathbb{K}(A_{f_i})$ for $1 \le i \le r$ so by Lemma 4.2(ii) there is a factorization in $\mathbb{K}(A)$ of the composite $M \longrightarrow P_M \longrightarrow X$ through a Koszul complex tensored with M

$$M \longrightarrow K(f_1^{B_1},\dots,f_r^{B_r}) \otimes M \xrightarrow{\psi} X \tag{4.9}$$

Since $M \longrightarrow P_M \longrightarrow X$ is nonzero the morphism ψ in (4.9) must also be nonzero. Setting $K = K(f_1^{B_1},\dots,f_r^{B_r})$ we know from Lemma 4.4 that $K \otimes P_M$ is a proper right projective resolution of $K \otimes M$, so there is an isomorphism

$$\operatorname{Hom}_{\mathbb{K}(\operatorname{Proj} A)}(K \otimes P_M, X) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}(A)}(K \otimes M, X)$$

We deduce that $\psi : K \otimes M \longrightarrow X$ factors through a nonzero morphism $K \otimes P_M \longrightarrow X$ in $\mathbb{K}(\operatorname{Proj} A)$, which provides the necessary morphism (4.7) and completes the proof. □

4.1 The Proof of Compact Generation

Given a noetherian scheme X we prove that $\mathbb{K}_m(\operatorname{Proj} X)$ is compactly generated by taking an affine open cover of X and "cocovering" the triangulated category $\mathbb{K}_m(\operatorname{Proj} X)$ by a family of Bousfield subcategories determined by the open cover. This reduces the problem to a question over affine schemes, where Proposition 4.5 of the previous section gives the input necessary to complete the proof.

Setup. In this section X is a fixed scheme and all sheaves are defined over X by default.

First we describe the triangulated subcategory of $\mathbb{K}_m(\operatorname{Proj} X)$ associated with an open subset of X, then we show that this subcategory is Bousfield.

Definition 4.6. Let $U \subseteq X$ be a quasi-compact open subset and write $\mathbb{K}_{m,X\setminus U}(\operatorname{Proj} X)$ for the kernel of the triangulated functor

$$(-)|_U : \mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{K}_m(\operatorname{Proj} U)$$

A complex $\mathscr{F}$ of flat quasi-coherent sheaves belongs to $\mathbb{K}_{m,X\setminus U}(\operatorname{Proj} X)$ if and only if the restriction $\mathscr{F}|_U$ is acyclic and $\mathbb{K}$-flat, in which case we say that $\mathscr{F}$ is *mock supported* on the closed set $X \setminus U$.

Lemma 4.7. *If $f : U \longrightarrow X$ is the inclusion of an affine open subset, then the functor $f_* : \mathbb{K}_m(\operatorname{Proj} U) \longrightarrow \mathbb{K}_m(\operatorname{Proj} X)$ is fully faithful and there is a localization sequence*

$$\begin{tikzcd} \mathbb{K}_{m,X\setminus U}(\operatorname{Proj} X) \arrow[r, shift left, "\mathrm{inc}"] & \mathbb{K}_m(\operatorname{Proj} X) \arrow[l, shift left] \arrow[r, shift left, "(-)|_U"] & \mathbb{K}_m(\operatorname{Proj} U) \arrow[l, shift left, "f_*"] \end{tikzcd}$$

Hence $\mathbb{K}_{m,X\setminus U}(\operatorname{Proj} X)$ is a Bousfield subcategory of $\mathbb{K}_m(\operatorname{Proj} X)$.

Proof. There is an adjunction between $(-)|_U$ and f_* on the level of the mock homotopy categories; see Definition 3.9. The counit $(-)|_U \circ f_* \longrightarrow 1$ is a natural equivalence, so a result of category theory tells us that the right adjoint f_* is fully faithful. As a consequence of Lemma 2.6 we have the desired localization sequence. □

These Bousfield subcategories determined by open subsets intersect properly with one another, in the sense of Rouquier; see [Rou03, Lemma 5.7] for the definition, which is also given in Chapter 2.

Lemma 4.8. *For quasi-compact open subsets $U, V \subseteq X$ the subcategories $\mathbb{K}_{m,X\setminus U}(\operatorname{Proj} X)$ and $\mathbb{K}_{m,X\setminus V}(\operatorname{Proj} X)$ are properly intersecting and Bousfield in $\mathbb{K}_m(\operatorname{Proj} X)$.*

Proof. To keep the notation light, set $\mathcal{T}_U = \mathbb{K}_{m,X\setminus U}(\operatorname{Proj} X)$ and $\mathcal{T}_V = \mathbb{K}_{m,X\setminus V}(\operatorname{Proj} X)$. First we treat the case where U and V are affine. By the previous lemma $\mathcal{T}_U$ is a Bousfield subcategory of $\mathbb{K}_m(\operatorname{Proj} X)$ and the functor $(-)|_U : \mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{K}_m(\operatorname{Proj} U)$ is a weak Verdier quotient, with the same statements holding for V. Let $f : U \longrightarrow X, g : V \longrightarrow X$ and $h : U \cap V \longrightarrow V$ be the inclusions, and observe that for $\mathscr{A}$ in $\mathcal{T}_V$ we have

$$f_*(\mathscr{A}|_U)|_V \cong h_*(\mathscr{A}|_{U\cap V}) = 0 \quad \text{ in } \mathbb{K}_m(\operatorname{Proj} V)$$

Hence $f_*(\mathscr{A}|_U)$ belongs to $\mathcal{T}_V$ and by symmetry $g_*(\mathscr{B}|_V)$ belongs to $\mathcal{T}_U$ for any $\mathscr{B}$ in $\mathcal{T}_U$. By Lemma 2.12, $\mathcal{T}_U$ and $\mathcal{T}_V$ are properly intersecting Bousfield subcategories of $\mathbb{K}_m(\operatorname{Proj} X)$.

Now let U be an arbitrary quasi-compact open subset of X, keeping V affine. Take an affine open cover $U = W_1 \cup \cdots \cup W_n$ and suppose that for some $1 \le i < n$ we have checked that the subcategory $\mathscr{S}_i = \mathbb{K}_{m,X\setminus(W_1\cup\cdots\cup W_i)}(\operatorname{Proj} X)$ is Bousfield and properly intersects every subcategory in the following list:

$$\mathbb{K}_{m,X\setminus W_{i+1}}(\operatorname{Proj} X),\ \dots\ ,\ \mathbb{K}_{m,X\setminus W_n}(\operatorname{Proj} X),\ \mathbb{K}_{m,X\setminus V}(\operatorname{Proj} X) \tag{4.10}$$

When $i = 1$ this is the affine case we have just verified. For arbitrary i, the subcategory $\mathscr{S}_{i+1}$ is the intersection of two properly intersecting Bousfield subcategories

$$\mathscr{S}_{i+1} = \mathscr{S}_i \cap \mathbb{K}_{m,X\setminus W_{i+1}}(\operatorname{Proj} X)$$

Using [Rou03, Lemma 5.8] and [Rou03, Lemma 5.9] we conclude that $\mathscr{S}_{i+1}$ is a Bousfield subcategory of $\mathbb{K}_m(\operatorname{Proj} X)$ intersecting properly with what remains of the list in (4.10) after we delete the first item. Proceeding inductively, we conclude that $\mathcal{T}_U$ is Bousfield and intersects properly with $\mathcal{T}_V$ for any affine open subset $V \subseteq X$. A similar argument on V now completes the proof. □

A *cocovering* of $\mathbb{K}_m(\operatorname{Proj} X)$ is a finite family of Bousfield subcategories, intersecting pairwise properly, with the intersection over all elements of the cover equal to zero.

Lemma 4.9. *Let $\mathfrak{U} = \{U_0, \dots, U_d\}$ be an affine open cover of X and set*

$$\mathcal{T}_i := \mathbb{K}_{m,X\setminus U_i}(\operatorname{Proj} X)$$

The family of Bousfield subcategories $\mathcal{F} = \{\mathcal{T}_0, \dots, \mathcal{T}_d\}$ is a cocovering of $\mathbb{K}_m(\operatorname{Proj} X)$.

Proof. By Lemma 4.8 the triangulated subcategories $\mathcal{T}_i$ are Bousfield subcategories that intersect properly with one another. An object of the intersection $\mathcal{T}_0 \cap \cdots \cap \mathcal{T}_d$ restricts to zero on an open cover of X, and is therefore by Remark 3.10 zero in $\mathbb{K}_m(\operatorname{Proj} X)$. Hence $\mathcal{F}$ is a cocovering of $\mathbb{K}_m(\operatorname{Proj} X)$. □

The next result is the major theorem of this chapter. The proof does not construct an explicit set of compact generators; we delay such a construction until Chapter 7.

Theorem 4.10. *If X is a noetherian scheme then $\mathbb{K}_m(\operatorname{Proj} X)$ is a compactly generated triangulated category and, for any open subset $U \subseteq X$, the subcategory $\mathbb{K}_{m,X\setminus U}(\operatorname{Proj} X)$ is compactly generated in $\mathbb{K}_m(\operatorname{Proj} X)$.*

Proof. Take an affine open cover $\mathfrak{U} = \{U_0, \dots, U_d\}$ of X and let $\mathcal{F}$ denote the associated cocovering by Bousfield subcategories defined in Lemma 4.9. To match our notation with the theorem of Rouquier (Theorem 2.13) that we will use, set $\mathcal{J} = \mathbb{K}_{m,X\setminus U}(\operatorname{Proj} X)$ and $\mathcal{T} = \mathbb{K}_m(\operatorname{Proj} X)$ for some open subset $U \subseteq X$, which may be empty. The proof consists of verifying the hypothesis in Rouquier's theorem.

Pick an arbitrary element $\mathcal{I}$ of the cocover $\mathcal{F}$. Reindexing if necessary, we may assume that the chosen element is $\mathcal{I} = \mathcal{T}_0$. For any subset $\mathcal{F}' \subseteq \{\mathcal{T}_1, \dots, \mathcal{T}_d\}$ we prove that the following quotient is compactly generated in $\mathcal{T}/\mathcal{I}$

$$\left(\mathcal{J} \cap \cap_{\mathcal{I}'\in\mathcal{F}'}\mathcal{I}'\right) / \left(\mathcal{J} \cap \cap_{\mathcal{I}'\in\mathcal{F}'\cup\{\mathcal{I}\}}\mathcal{I}'\right) \tag{4.11}$$

Let $W \subseteq X$ be the union of U with all those open sets U_i with $\mathcal{T}_i \in \mathcal{F}'$. Since $\mathcal{J} \cap (\cap_{\mathcal{I}' \in \mathcal{F}'} \mathcal{I}')$ is the category $\mathbb{K}_{m,X \setminus W}(\operatorname{Proj} X)$, we have to prove that the essential image of the composite

$$\mathbb{K}_{m,X\setminus W}(\operatorname{Proj} X) \xrightarrow{\text{inc}} \mathbb{K}_m(\operatorname{Proj} X) \xrightarrow{\text{can}} \mathbb{K}_m(\operatorname{Proj} X)/\,\mathbb{K}_{m,X\setminus U_0}(\operatorname{Proj} X)$$

is compactly generated in $\mathbb{K}_m(\operatorname{Proj} X)/\,\mathbb{K}_{m,X\setminus U_0}(\operatorname{Proj} X)$. This quotient is equivalent to $\mathbb{K}_m(\operatorname{Proj} U_0)$ by Lemma 4.7, so it is enough to show that the essential image of the following composite (call the essential image $\mathcal{S}$) is compactly generated in $\mathbb{K}_m(\operatorname{Proj} U_0)$

$$\mathbb{K}_{m,X\setminus W}(\operatorname{Proj} X) \xrightarrow{\text{inc}} \mathbb{K}_m(\operatorname{Proj} X) \xrightarrow{(-)|_{U_0}} \mathbb{K}_m(\operatorname{Proj} U_0)$$

We claim that the image $\mathcal{S}$ is the kernel of $(-)|_{U_0 \cap W} : \mathbb{K}_m(\operatorname{Proj} U_0) \longrightarrow \mathbb{K}_m(\operatorname{Proj} U_0 \cap W)$. It is clear that $\mathcal{S}$ is contained in the kernel. To prove the reverse inclusion, let a complex $\mathscr{F}$ in the kernel be given, write $f : U_0 \longrightarrow X, h : U_0 \cap W \longrightarrow W$ for the inclusions, and set $\mathscr{F}' = f_*(\mathscr{F})$. Then $\mathscr{F}'|_W = h_*(\mathscr{F}|_{U_0 \cap W})$ is zero in $\mathbb{K}_m(\operatorname{Proj} W)$ and $\mathscr{F} = \mathscr{F}'|_{U_0}$, hence $\mathscr{F}$ is an object of $\mathcal{S}$, which proves the claim.

Because U_0 is affine there exists a noetherian ring A and an isomorphism of schemes $U_0 \cong \mathit{Spec}(A)$, which identifies $U_0 \cap W$ with a finite union $D(f_1) \cup \cdots \cup D(f_r)$ for some elements $f_1, \ldots, f_r \in A$, where $D(f_j)$ is the open set of prime ideals not containing f_j. The kernel of the restriction functor

$$(-)|_{U_0 \cap W} : \mathbb{K}_m(\operatorname{Proj} U_0) \longrightarrow \mathbb{K}_m(\operatorname{Proj} U_0 \cap W)$$

corresponds under the equivalence $\mathbb{K}(\operatorname{Proj} A) \cong \mathbb{K}_m(\operatorname{Proj} U_0)$ to the subcategory

$$\mathcal{V}(f_1, \ldots, f_r) = \{P \in \mathbb{K}(\operatorname{Proj} A) \,|\, P_{f_i} \text{ is zero in } \mathbb{K}(\operatorname{Proj} A_{f_i}) \text{ for } 1 \le i \le r\}$$

We proved in Proposition 4.5 that $\mathcal{V}(f_1, \ldots, f_r)$ is compactly generated in $\mathbb{K}(\operatorname{Proj} A)$, from which it follows that $\mathcal{S}$ is compactly generated in $\mathbb{K}_m(\operatorname{Proj} U_0)$. This completes the proof that (4.11) is compactly generated in $\mathcal{T}/\mathcal{I}$.

If we take U to be empty in the above, then $\mathbb{K}_{m,X\setminus U}(\operatorname{Proj} X)$ is just $\mathbb{K}_m(\operatorname{Proj} X)$ and Theorem 2.13 allows us to conclude that $\mathbb{K}_m(\operatorname{Proj} X)$ is compactly generated. Given an arbitrary open subset $U \subseteq X$ another application of Theorem 2.13 (using Lemma 4.8 to check the hypothesis about proper intersection) proves that $\mathbb{K}_{m,X\setminus U}(\operatorname{Proj} X)$ is compactly generated in $\mathbb{K}_m(\operatorname{Proj} X)$, and completes the proof. $\square$

Chapter 5

The Mock Stable Derived Category

Let X be a scheme and let $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ denote the full subcategory of acyclic complexes in $\mathbb{K}_m(\operatorname{Proj} X)$. This triangulated category, called the *mock stable derived category* of X, will turn out to be an invariant of the singularities of X (see Section 9.2). Our main result in this chapter asserts that there is a recollement (Theorem 5.5)

$$\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \leftrightarrows \mathbb{K}_m(\operatorname{Proj} X) \leftrightarrows \mathbb{D}(\mathfrak{Qco} X)$$

from which we deduce that $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ is compactly generated when X is noetherian. Here $\mathbb{D}(\mathfrak{Qco} X)$ is the derived category of quasi-coherent sheaves on X, and the recollement adjoins to this derived category the mock stable derived category, whose objects are acyclic complexes of flat quasi-coherent sheaves (we review recollements in Chapter 2).

The recollement says something about $\mathbb{K}$-flat resolutions: using it, we prove in Remark 5.9 that $\mathbb{K}$-flat resolutions are unique in $\mathbb{K}_m(\operatorname{Proj} X)$. Together with the closed monoidal structure studied in the next chapter, this observation will lead to our characterization in Chapter 7 of the compact objects in $\mathbb{K}_m(\operatorname{Proj} X)$. Specializing to an affine scheme, there a recollement for any ring A

$$\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A) \leftrightarrows \mathbb{K}(\operatorname{Proj} A) \leftrightarrows \mathbb{D}(A)$$

which exists even for noncommutative rings (Theorem 5.15). It follows that the homotopy category $\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ of acyclic complexes of projective modules is always well generated, and that it is compactly generated whenever A is right coherent (Corollary 5.17).

Setup. In this chapter X is a fixed scheme and all sheaves are defined over X by default.

Before embarking on the proofs, let us explain the connection between stable module categories and the triangulated category $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$. In the same way that the mock homotopy category $\mathbb{K}_m(\operatorname{Proj} X)$ glues together the homotopy categories $\mathbb{K}(\operatorname{Proj} A)$ over affine open subsets, the mock stable derived category glues together homotopy categories $\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ of acyclic complexes of projectives, so it is worth talking a little bit about acyclic complexes of projective modules over a ring A. Let P be such an acyclic complex, and consider the kernel (also called the *syzygy*) $K = Ker(P^n \longrightarrow P^{n+1})$ in degree n

$$\begin{tikzcd}
\cdots \arrow[r] & P^{n-2} \arrow[r] & P^{n-1} \arrow[rr] \arrow[dr, bend right] & & P^{n} \arrow[r] & P^{n+1} \arrow[r] & \cdots \\
& & & K \arrow[ur, bend right] & & &
\end{tikzcd}$$

The truncation $\cdots \longrightarrow P^{n-2} \longrightarrow P^{n-1}$ is a projective resolution of K so, up to homotopy equivalence, P is determined in degrees $\leq n-1$ by K. In some cases the entire complex is determined by this syzygy. To give a noncommutative example, if $A = kG$ is the group ring of a field k and finite group G then the projective and injective A-modules coincide, and in the above diagram P is an injective resolution in degrees $\geq n$. The complex P is determined, up to homotopy equivalence, by K. The best way to describe this correspondence between acyclic complexes and syzygies is via stable module categories.

There is a generic notion of *stabilization* in algebra. Abstractly, this is the process of modding out the "free" or "smooth" parts of the representation theory of some structure. The *stable module category* of a (noncommutative) ring A, denoted $\underline{\mathbf{Mod}}A$, has the same objects as the ordinary module category but morphisms are identified if their difference factors through a projective. This category does not see projective summands of a module; it sees only the "hard" part of the representation theory of A. For some entry points into the literature, see the work of Auslander and Bridger [AB69], Happel's book [Hap88], the papers of Rickard [Ric89, Ric97], Benson-Carlson-Rickard [BCR97] and Rouquier [CR00].

A canonical example is the stable module category $\underline{\mathbf{Mod}}kG$ of a field k and finite group G. Over the group ring $A = kG$ a module is projective if and only if it is injective, and acyclic complexes of projective modules are in bijection with modules. To be precise, there is an equivalence of triangulated categories [Kra05, Theorem 8.2]

$$Z^0(-) : \mathbb{K}_{\mathrm{ac}}(\mathrm{Inj}\, A) = \mathbb{K}_{\mathrm{ac}}(\mathrm{Proj}\, A) \xrightarrow{\sim} \underline{\mathbf{Mod}}A \tag{5.1}$$

defined by taking the kernel in degree zero; see [Kra05, §8] for further details. However, we are primarily interested in the commutative rings occurring in algebraic geometry, and for such rings the injective and projective modules rarely agree. To rescue the correspondence between acyclic complexes and their syzygies we restrict to the acyclic complexes for which the correspondence survives, called the *totally acyclic* complexes.

Rather than define total acyclicity, let us work over a (commutative) Gorenstein ring A of finite Krull dimension, where acyclic complexes are totally acyclic [IK06, Corollary 5.5]. Even then, not every module occurs as a syzygy of an acyclic complex of projectives; those that do are called *Gorenstein projective* modules, and we denote by $\mathrm{Gproj}(\underline{\mathbf{Mod}}A)$ the full subcategory of $\underline{\mathbf{Mod}}A$ consisting of these modules. Having arranged things correctly, we have the analogue of (5.1): there is an equivalence of triangulated categories (Proposition 5.10) defined by taking the kernel in degree zero

$$Z^0(-) : \mathbb{K}_{\mathrm{ac}}(\mathrm{Proj}\, A) \xrightarrow{\sim} \mathrm{Gproj}(\underline{\mathbf{Mod}}A)$$

The compact objects on the right hand side are, up to direct factors, the finitely generated Gorenstein projective modules, which agree in this case with the *maximal Cohen-Macaulay (MCM) modules*; see Lemma 5.11. The stable module category of MCM modules contains information about the singularities of the ring; see for example [Yos90]. This completes our local description of the mock stable derived category: over a finite dimensional Gorenstein scheme, the triangulated category $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ is locally equivalent to the stable module category $\operatorname{Gproj}(\underline{\mathbf{Mod}}A)$ which contains, in its subcategory of compact objects, the stable module category of MCM modules. Results of this type explain how $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ carries information about the singularities of X.

The theory of homotopy categories that we are describing comes to us in two flavours: projective and injective. In this chapter we study the projective version $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ of the stable derived category, but the injective aspect was developed first, by Krause. In [Kra05] he introduced the (injective) stable derived category $\mathbb{K}_{\mathrm{ac}}(\operatorname{Inj} X)$ of a noetherian scheme and proved many interesting results about it, including the existence of a recollement whose projective analogue the reader will encounter in Theorem 5.5 below.

Having explained the motivation, let us proceed with the results. The first proposition tells us that the orthogonal ${}^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ in $\mathbb{K}_m(\operatorname{Proj} X)$ is the subcategory of $\mathbb{K}$-flat complexes (the definition of $\mathbb{K}$-flatness in given in Section 2.2). But first we need to check that being $\mathbb{K}$-flat is a property stable under isomorphism in $\mathbb{K}_m(\operatorname{Proj} X)$.

Lemma 5.1. *Let $\mathscr{F}, \mathscr{G}$ be two complexes of flat quasi-coherent sheaves isomorphic in $\mathbb{K}_m(\operatorname{Proj} X)$. Then $\mathscr{G}$ is $\mathbb{K}$-flat if and only if $\mathscr{F}$ is $\mathbb{K}$-flat.*

Proof. Assume that $\mathscr{G}$ is $\mathbb{K}$-flat and let $Q : \mathbb{K}(\operatorname{Flat} X) \longrightarrow \mathbb{K}_m(\operatorname{Proj} X)$ denote the quotient functor. Every morphism $\alpha : \mathscr{F} \longrightarrow \mathscr{G}$ in $\mathbb{K}_m(\operatorname{Proj} X)$ can be written as $Q(b)Q(a)^{-1}$ for morphisms $a : \mathscr{W} \longrightarrow \mathscr{F}$ and $b : \mathscr{W} \longrightarrow \mathscr{G}$ in $\mathbb{K}(\operatorname{Flat} X)$ with a having mapping cone in $\mathbb{E}(X)$. If α is an isomorphism in $\mathbb{K}_m(\operatorname{Proj} X)$ then b must also be an isomorphism, and we deduce that b has mapping cone in $\mathbb{E}(X)$ as a morphism of $\mathbb{K}(\operatorname{Flat} X)$. Extending b to a triangle in $\mathbb{K}(\operatorname{Flat} X)$ we have

$$\mathscr{W} \xrightarrow{b} \mathscr{G} \longrightarrow \mathscr{T} \longrightarrow \Sigma\mathscr{W}$$

where $\mathscr{T}$ and $\mathscr{G}$ are both $\mathbb{K}$-flat, because complexes in $\mathbb{E}(X)$ are $\mathbb{K}$-flat. We deduce that $\mathscr{W}$ is $\mathbb{K}$-flat. Extending a to a triangle $\mathscr{W} \longrightarrow \mathscr{F} \longrightarrow \mathscr{T}' \longrightarrow \Sigma\mathscr{W}$ we have $\mathscr{T}'$ in $\mathbb{E}(X)$ by construction, so both $\mathscr{T}'$ and $\mathscr{W}$ are $\mathbb{K}$-flat. We conclude that $\mathscr{F}$ is $\mathbb{K}$-flat, as required. ☐

It is a basic fact of homological algebra that flat modules are close to projective modules, despite the former being defined in terms of tensor products and the latter in terms of Hom. The generalization of these notions to complexes ($\mathbb{K}$-projectivity and $\mathbb{K}$-flatness) follows the same pattern; see Section 2.2. This makes the following characterization of $\mathbb{K}$-flatness slightly surprising; informally, passing to the mock homotopy category has the effect of making $\mathbb{K}$-flat complexes behave like $\mathbb{K}$-projective complexes.

Proposition 5.2. *A complex $\mathscr{F}$ of flat quasi-coherent sheaves is $\mathbb{K}$-flat if and only if it belongs to the orthogonal ${}^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ as an object of $\mathbb{K}_m(\operatorname{Proj} X)$.*

Proof. First we prove the claim when the complex $\mathscr{F}$ is bounded above. In this case $\mathscr{F}$ is already $\mathbb{K}$-flat, so we have to show that it belongs to the orthogonal ${}^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$.

This is a local question: suppose that we know it is true for affine schemes, and take an affine open cover $\{U_0, \ldots, U_d\}$ of X. Then, by assumption, for any finite intersection V of open sets in the cover the restriction $\mathscr{F}|_V$ belongs to the orthogonal ${}^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} V)$. Denoting the inclusion by $f : V \longrightarrow X$ we have, for any acyclic complex $\mathscr{C}$

$$\operatorname{Hom}_{\mathbb{K}_m(\operatorname{Proj} X)}(\mathscr{F}, f_*(\mathscr{C}|_V)) \cong \operatorname{Hom}_{\mathbb{K}_m(\operatorname{Proj} V)}(\mathscr{F}|_V, \mathscr{C}|_V) = 0$$

Setting $\mathcal{L} = \{\mathscr{F}\}^{\perp}$ in Corollary 3.14 we conclude that $\mathscr{F}$ is left orthogonal in $\mathbb{K}_m(\operatorname{Proj} X)$ to acyclic complexes. Therefore, to prove that every bounded above complex $\mathscr{F}$ belongs to ${}^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$, we can reduce to the affine case where $X = \mathit{Spec}(A)$ and $\mathscr{F} = F^{\sim}$ for a bounded above complex F of flat A-modules. In this situation, let $P \longrightarrow F$ be a quasi-isomorphism with P a bounded above complex of projective A-modules. Extending to a triangle in $\mathbb{K}(\operatorname{Flat} A)$

$$P \longrightarrow F \longrightarrow E \longrightarrow \Sigma P$$

the mapping cone E is an acyclic, bounded above complex of flat modules. Such complexes belong to $\mathbb{E}(A)$ so $P \longrightarrow F$ is an isomorphism in $\mathbb{K}_m(\operatorname{Proj} A)$. We want to show that F is left orthogonal to acyclic complexes. Up to isomorphism every complex in $\mathbb{K}_m(\operatorname{Proj} A)$ is a complex of projectives (Lemma 3.6) so it suffices to check that F is left orthogonal to every acyclic complex Z of projective modules. In this case

$$\operatorname{Hom}_{\mathbb{K}_m(\operatorname{Proj} A)}(F, Z) \cong \operatorname{Hom}_{\mathbb{K}_m(\operatorname{Proj} A)}(P, Z) \cong \operatorname{Hom}_{\mathbb{K}(\operatorname{Proj} A)}(P, Z) = 0$$

since P is $\mathbb{K}$-projective and Z is acyclic. This completes the proof of the proposition when $\mathscr{F}$ is a bounded above complex. For an arbitrary complex $\mathscr{F} \in \mathbb{K}_m(\operatorname{Proj} X)$ we can by Corollary 3.22 find a sequence of bounded above complexes of flat quasi-coherent sheaves

$$\mathscr{P}_0 \longrightarrow \mathscr{P}_1 \longrightarrow \mathscr{P}_2 \longrightarrow \mathscr{P}_3 \longrightarrow \cdots$$

whose homotopy colimit in $\mathbb{K}(\operatorname{Flat} X)$ is a $\mathbb{K}$-flat complex $\mathscr{P}$ quasi-isomorphic to $\mathscr{F}$. The localizing subcategory ${}^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ contains the $\mathscr{P}_i$ by the above discussion, and thus contains the homotopy colimit $\mathscr{P}$. We have a triangle in $\mathbb{K}(\operatorname{Flat} X)$ with $\mathscr{C}$ acyclic

$$\mathscr{P} \longrightarrow \mathscr{F} \longrightarrow \mathscr{C} \longrightarrow \Sigma\mathscr{P}$$

Using this triangle we make the following chain of deductions

$$\begin{aligned}
\mathscr{F} \text{ is } \mathbb{K}\text{-flat} &\iff \mathscr{C} \text{ is } \mathbb{K}\text{-flat, since } \mathscr{P} \text{ is known to be } \mathbb{K}\text{-flat} \\
&\iff \mathscr{C} \text{ is zero in } \mathbb{K}_m(\operatorname{Proj} X)\text{, since it is already acyclic} \\
&\iff \mathscr{C} \text{ belongs to } {}^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \\
&\iff \mathscr{F} \text{ belongs to } {}^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)
\end{aligned}$$

which is what we needed to show. □

Definition 5.3. Let $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ denote the full subcategory of acyclic complexes in $\mathbb{K}_m(\operatorname{Proj} X)$. The canonical functor $\mathbb{K}(\operatorname{Flat} X) \longrightarrow \mathbb{D}(\mathfrak{Qco} X)$ vanishes on complexes in $\mathbb{E}(X)$, so there is a unique triangulated functor U making the following diagram commute

$$\begin{tikzcd}
\mathbb{K}(\operatorname{Flat} X) \arrow[r, "\mathrm{inc}"] \arrow[dr, bend right, "Q"'] & \mathbb{K}(\mathfrak{Qco} X) \arrow[r, "\mathrm{can}"] & \mathbb{D}(\mathfrak{Qco} X) \\
& \mathbb{K}_m(\operatorname{Proj} X) \arrow[ur, bend right, "U"'] &
\end{tikzcd}$$

where Q is the Verdier quotient. Clearly $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ is the kernel of U, and in particular it is a localizing subcategory of $\mathbb{K}_m(\operatorname{Proj} X)$.

Unsurprisingly, the proof of the major theorem in this chapter is easier for noetherian schemes. The reader willing to assume noetherianness can skip the next result, which is superfluous in the noetherian case.

Proposition 5.4. *The inclusion*

$$\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \longrightarrow \mathbb{K}_m(\operatorname{Proj} X)$$

has a right adjoint.

Proof. The proof is by reduction to the affine case, which is true even for noncommutative rings so we delay it until Section 5.1; see Theorem 5.15. Assuming the affine case, the proof follows the now familiar pattern of a Čech argument (cf. the proof of Theorem 3.16). Let $\mathcal{L}$ be the full subcategory of $\mathbb{K}_m(\operatorname{Proj} X)$ consisting of the complexes $\mathscr{F}$ that fit into a triangle in $\mathbb{K}_m(\operatorname{Proj} X)$

$$\mathscr{C} \longrightarrow \mathscr{F} \longrightarrow \mathscr{S} \longrightarrow \Sigma\mathscr{C} \tag{5.2}$$

with $\mathscr{C}$ belonging to $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ and $\mathscr{S}$ in $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)^{\perp}$. This subcategory $\mathcal{L}$ is a triangulated subcategory called the *Verdier sum*. We claim that $\mathcal{L} = \mathbb{K}_m(\operatorname{Proj} X)$.

To prove this claim it suffices, using Corollary 3.14, to show that $f_*(\mathscr{G})$ belongs to $\mathcal{L}$ for every affine open subset $V \subseteq X$ and $\mathscr{G}$ in $\mathbb{K}_m(\operatorname{Proj} V)$, where $f : V \longrightarrow X$ denotes the inclusion. But the result is true for affine schemes by Theorem 5.15, so we have a triangle in $\mathbb{K}_m(\operatorname{Proj} V)$ with $\mathscr{C}$ in $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} V)$ and $\mathscr{S}$ in the orthogonal $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} V)^{\perp}$

$$\mathscr{C} \longrightarrow \mathscr{G} \longrightarrow \mathscr{S} \longrightarrow \Sigma\mathscr{C}$$

Applying the functor $f_* : \mathbb{K}_m(\operatorname{Proj} V) \longrightarrow \mathbb{K}_m(\operatorname{Proj} X)$ we have a triangle in $\mathbb{K}_m(\operatorname{Proj} X)$

$$f_*(\mathscr{C}) \longrightarrow f_*(\mathscr{G}) \longrightarrow f_*(\mathscr{S}) \longrightarrow \Sigma f_*(\mathscr{C})$$

where $f_*(\mathscr{C})$ is acyclic, because f is affine, and $f_*(\mathscr{S})$ belongs to $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)^{\perp}$ because of the adjunction between direct image and restriction. We conclude that every complex in $\mathbb{K}_m(\operatorname{Proj} X)$ fits into a triangle (5.2) of the desired form, which is enough by Lemma 2.3 to prove the existence of the required adjoint. $\square$

We are now ready for the proof of the theorem.

Theorem 5.5. *The sequence*

$$\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \xrightarrow{\mathrm{inc}} \mathbb{K}_m(\operatorname{Proj} X) \xrightarrow{U} \mathbb{D}(\mathfrak{Qco}X) \tag{5.3}$$

induces a recollement

$$\begin{tikzcd} \mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] & \mathbb{K}_m(\operatorname{Proj} X) \arrow[r, shift left=2, leftarrow] \arrow[r] \arrow[r, shift right=2, leftarrow] & \mathbb{D}(\mathfrak{Qco}X) \end{tikzcd} \tag{5.4}$$

It follows that $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ *is compactly generated when* X *is noetherian.*

Proof. Let us outline the proof: Lemma 2.6 tells us that any triangulated functor with a fully faithful left adjoint is, up to equivalence, the Verdier quotient by its kernel. We want to prove that U has this property. It will be enough to show that the composite

$$T : {}^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \longrightarrow \mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{D}(\mathfrak{Qco}X) \tag{5.5}$$

is an equivalence, which follows from our identification in Proposition 5.2 of the orthogonal ${}^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ with the subcategory of $\mathbb{K}$-flat complexes. This will prove that (5.3) is a colocalization sequence; we then invoke Proposition 5.4 to see that it is a recollement.

Now for the details. In order to prove that U has a fully faithful left adjoint, we show that for complexes $\mathscr{P}$ and $\mathscr{F}$ in $\mathbb{K}_m(\operatorname{Proj} X)$ with $\mathscr{P}$ a $\mathbb{K}$-flat complex, the map induced by the canonical functor U

$$\Phi : \operatorname{Hom}_{\mathbb{K}_m(\operatorname{Proj} X)}(\mathscr{P}, \mathscr{F}) \longrightarrow \operatorname{Hom}_{\mathbb{D}(\mathfrak{Qco}X)}(\mathscr{P}, \mathscr{F})$$

is a bijection. The following simple observation is the crux of the argument:

> $(\star)$ Let $\mathscr{P}' \longrightarrow \mathscr{P}$ be a quasi-isomorphism of $\mathbb{K}$-flat complexes of flat quasi-coherent sheaves. The mapping cone is acyclic and $\mathbb{K}$-flat, and thus vanishes in $\mathbb{K}_m(\operatorname{Proj} X)$, so $\mathscr{P}' \longrightarrow \mathscr{P}$ is an isomorphism in $\mathbb{K}_m(\operatorname{Proj} X)$.

To check surjectivity of Φ let a morphism $\alpha : \mathscr{P} \longrightarrow \mathscr{F}$ in $\mathbb{D}(\mathfrak{Qco}X)$ be given. This can be represented by a "roof" diagram in $\mathbb{K}(\mathfrak{Qco}X)$

$$\begin{tikzcd} & \mathscr{W} \arrow[dl, "a"'] \arrow[dr, "b"] & \\ \mathscr{P} & & \mathscr{F} \end{tikzcd}$$

where a is a quasi-isomorphism. Denoting the quotient by $q : \mathbb{K}(\mathfrak{Qco}X) \longrightarrow \mathbb{D}(\mathfrak{Qco}X)$ this means that $\alpha = q(b)q(a)^{-1}$ in $\mathbb{D}(\mathfrak{Qco}X)$. Using Corollary 3.22 we can, by replacing $\mathscr{W}$ with its resolution if necessary, assume that $\mathscr{W}$ is a $\mathbb{K}$-flat complex of flat quasi-coherent sheaves. It now follows from $(\star)$ that a is an isomorphism in $\mathbb{K}_m(\operatorname{Proj} X)$, so $\alpha = \Phi(Q(b)Q(a)^{-1})$ is in the image of Φ, where $Q : \mathbb{K}(\operatorname{Flat} X) \longrightarrow \mathbb{K}_m(\operatorname{Proj} X)$ is the quotient.

To see that Φ is injective, it suffices to show that a morphism of *complexes* $s : \mathscr{P} \longrightarrow \mathscr{F}$ sent to zero in $\mathbb{D}(\mathfrak{Qco}X)$ is already zero in $\mathbb{K}_m(\operatorname{Proj} X)$. If s is zero in $\mathbb{D}(\mathfrak{Qco}X)$ then by [Nee01b, Lemma 2.1.26] we can find a quasi-isomorphism $t : \mathscr{W} \longrightarrow \mathscr{P}$ of complexes of quasi-coherent sheaves with the following composite zero in $\mathbb{K}(\mathfrak{Qco}X)$

$$\mathscr{W} \xrightarrow{t} \mathscr{P} \xrightarrow{s} \mathscr{F}$$

Let $t' : \mathscr{P}' \longrightarrow \mathscr{W}$ be a $\mathbb{K}$-flat resolution of $\mathscr{W}$ by a complex $\mathscr{P}'$ of flat quasi-coherent sheaves. The composite $t \circ t'$ is a quasi-isomorphism $\mathscr{P}' \longrightarrow \mathscr{P}$ of $\mathbb{K}$-flat complexes, therefore an isomorphism in $\mathbb{K}_m(\operatorname{Proj} X)$ by $(\star)$. Since s composes with this morphism to give zero in $\mathbb{K}_m(\operatorname{Proj} X)$, we deduce that s is zero in $\mathbb{K}_m(\operatorname{Proj} X)$, as required. This completes the proof that Φ is a bijection, and shows that T in (5.5) is fully faithful.

The functor T is an equivalence, because every object of $\mathbb{D}(\mathfrak{Qco} X)$ admits a $\mathbb{K}$-flat resolution by flat quasi-coherent sheaves (Corollary 3.22) and these resolutions are objects of ${}^{\perp}\,\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ (Proposition 5.2). Composing the quasi-inverse T^{-1} with the inclusion of the orthogonal ${}^{\perp}\,\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \longrightarrow \mathbb{K}_m(\operatorname{Proj} X)$ gives a fully faithful left adjoint for U, so it follows from Lemma 2.6 that (5.3) is a colocalization sequence.

When X is noetherian, $\mathbb{K}_m(\operatorname{Proj} X)$ is compactly generated and U has a right adjoint, because it preserves coproducts (Proposition 2.7). More generally, Proposition 5.4 ensures the existence of a right adjoint to the inclusion $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \longrightarrow \mathbb{K}_m(\operatorname{Proj} X)$ for any scheme. This gives two different proofs, via Lemma 2.3, that (5.3) is a recollement. Finally, when X is noetherian the category $\mathbb{K}_m(\operatorname{Proj} X)$ is compactly generated (Theorem 4.10) so it follows from Corollary 2.10 that $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ is also compactly generated. $\square$

A weaker version of the theorem holds for $\mathbb{K}(\operatorname{Flat} X)$.

Corollary 5.6. *There is a localization sequence*

$$\mathbb{K}_{\mathrm{ac}}(\operatorname{Flat} X) \rightleftarrows \mathbb{K}(\operatorname{Flat} X) \rightleftarrows \mathbb{D}(\mathfrak{Qco} X)$$

Proof. It is enough by Lemma 2.6 to prove that $\mathbb{K}(\operatorname{Flat} X) \longrightarrow \mathbb{D}(\mathfrak{Qco} X)$ admits a fully faithful right adjoint. This functor is the composite of $\mathbb{K}(\operatorname{Flat} X) \longrightarrow \mathbb{K}_m(\operatorname{Proj} X)$ and $\mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{D}(\mathfrak{Qco} X)$, which by Theorem 3.16 and Theorem 5.5 both have fully faithful right adjoints. Hence so does their composite. $\square$

Remark 5.7. The localization sequence of the corollary is a recollement when X is affine, see Corollary 5.16 below, but this is not true over general schemes. For a counterexample, see Remark A.15.

Remark 5.8. In particular, we have equivalences of triangulated categories

$$\mathbb{K}_m(\operatorname{Proj} X)/\,\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \xrightarrow{\sim} \mathbb{D}(\mathfrak{Qco} X), \quad \mathbb{K}(\operatorname{Flat} X)/\,\mathbb{K}_{\mathrm{ac}}(\operatorname{Flat} X) \xrightarrow{\sim} \mathbb{D}(\mathfrak{Qco} X)$$

Remark 5.9 (Adjoints and flat resolutions)**.** The existence of $\mathbb{K}$-injective and $\mathbb{K}$-projective resolutions can be phrased as the existence of a recollement (see Remark 2.20) involving the homotopy category and derived category of an abelian category $\mathcal{A}$

$$\mathbb{K}_{\mathrm{ac}}(\mathcal{A}) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathbb{K}(\mathcal{A}) \;\substack{\longleftarrow\\ \longrightarrow\\ \longleftarrow}\; \mathbb{D}(\mathcal{A}) \tag{5.6}$$

When $\mathcal{A}$ is the category of modules over a ring this recollement exists, and the left adjoint q_λ of the quotient $q : \mathbb{K}(\mathcal{A}) \longrightarrow \mathbb{D}(\mathcal{A})$ sends a complex to its $\mathbb{K}$-projective resolution. Given an arbitrary Grothendieck abelian category, for example the category of quasi-coherent sheaves over a scheme, (5.6) is a localization sequence but not necessarily a recollement, because there may not be enough projectives.

For an abelian category $\mathcal{A}$ that lacks projectives there may be a weaker notion of a *flat* object, and in many examples such flat objects are plentiful. Theorem 5.5 tells us that by passing to the mock homotopy category we can make flats imitate projectives:

(i) *Flat resolutions are unique up to isomorphism in* $\mathbb{K}_m(\operatorname{Proj} X)$. It follows from the theorem that we have an equivalence

$$^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \xrightarrow{\mathrm{inc}} \mathbb{K}_m(\operatorname{Proj} X) \xrightarrow{\mathrm{can}} \mathbb{D}(\mathfrak{Qco}X) \tag{5.7}$$

The left adjoint U_λ factors as an equivalence $\mathbb{D}(\mathfrak{Qco}X) \xrightarrow{\sim} {}^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ followed by the inclusion, and this defines the quasi-inverse of (5.7). Recall from Proposition 5.2 that the orthogonal ${}^{\perp}\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} X)$ is the full subcategory of $\mathbb{K}$-flat complexes.

We infer that $\mathbb{K}$-flat complexes are isomorphic in $\mathbb{K}_m(\operatorname{Proj} X)$ if and only if they are isomorphic in $\mathbb{D}(\mathfrak{Qco}X)$. Hence $\mathbb{K}$-flat resolutions by flat quasi-coherent sheaves are unique up to isomorphism; note that such resolutions exist, by Corollary 3.22.

(ii) *Flat resolutions are functorial in* $\mathbb{K}_m(\operatorname{Proj} X)$. Let $\mathscr{M} \longrightarrow \mathscr{N}$ be a morphism in $\mathbb{D}(\mathfrak{Qco}X)$ and choose $\mathbb{K}$-flat resolutions $\mathscr{P}_{\mathscr{M}} \longrightarrow \mathscr{M}$ and $\mathscr{P}_{\mathscr{N}} \longrightarrow \mathscr{N}$ consisting of flat quasi-coherent sheaves. We have a diagram in $\mathbb{D}(\mathfrak{Qco}X)$

$$\begin{tikzcd} \mathscr{P}_{\mathscr{M}} \arrow[r, "\sim"] \arrow[d, dotted] & \mathscr{M} \arrow[d] \\ \mathscr{P}_{\mathscr{N}} \arrow[r, "\sim"'] & \mathscr{N} \end{tikzcd} \tag{5.8}$$

The composite $\mathscr{P}_{\mathscr{M}} \xrightarrow{\sim} \mathscr{M} \longrightarrow \mathscr{N} \xrightarrow{\sim} \mathscr{P}_{\mathscr{N}}$ in $\mathbb{D}(\mathfrak{Qco}X)$ lifts by (5.7) to a unique morphism $\mathscr{P}_{\mathscr{M}} \longrightarrow \mathscr{P}_{\mathscr{N}}$ in $\mathbb{K}_m(\operatorname{Proj} X)$ making (5.8) into a commutative square in $\mathbb{D}(\mathfrak{Qco}X)$. This demonstrates the functoriality of flat resolutions.

Let us retreat for a moment from sweeping generality and construct this lifting in the affine case. Let A be a ring and $f : M \longrightarrow N$ a morphism of A-modules. Let F_M, F_N denote flat resolutions of M and N, respectively, and choose a projective resolution P of M. In the standard way, we construct morphisms of complexes $\alpha : P \longrightarrow F_M$ and $\beta : P \longrightarrow F_N$ lifting the identity and f, respectively, as in the following diagram

$$\begin{tikzcd} & \cdots \arrow[r] & F_M^{-1} \arrow[r] & F_M^0 \arrow[r] & M \\ \cdots \arrow[r] & P^{-1} \arrow[r] \arrow[ur] \arrow[dr] & P^0 \arrow[r] \arrow[ur] \arrow[dr] & M \arrow[ur, "1"] \arrow[dr, "f"'] & \\ & \cdots \arrow[r] & F_N^{-1} \arrow[r] & F_N^0 \arrow[r] & N \end{tikzcd}$$

We already observed in Remark 3.7 that α is a morphism with mapping cone in $\mathbb{E}(A)$, that is, it is an isomorphism in $\mathbb{K}_m(\operatorname{Proj} A)$. The composite $\varphi = \beta \circ \alpha^{-1}$ is a morphism $F_M \longrightarrow F_N$ in $\mathbb{K}_m(\operatorname{Proj} A)$ that lifts f to the flat resolutions.

(iii) *Flat resolutions are calculated by the left adjoint of U.* Let $\mathscr{M}$ be a complex of quasi-coherent sheaves and $\mathscr{P}_{\mathscr{M}} \longrightarrow \mathscr{M}$ a $\mathbb{K}$-flat resolution by flat quasi-coherent sheaves. As $\mathscr{P}_{\mathscr{M}}$ belongs to the orthogonal ${}^{\perp}\mathbb{K}_m(\operatorname{Proj} X)$ we have an isomorphism

$$U_\lambda(\mathscr{M}) \cong U_\lambda(\mathscr{P}_{\mathscr{M}}) \cong \mathscr{P}_{\mathscr{M}}$$

in $\mathbb{K}_m(\operatorname{Proj} X)$, so U_λ sends a complex to its $\mathbb{K}$-flat resolution.

We conclude this section by giving a local description of $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ in an important special case, proving the claim made in the introduction about stable module categories of Gorenstein projective modules. But first, let us set up some notation.

Let A be a noetherian ring. An acyclic complex P of projective A-modules is *totally acyclic* if the complex $\operatorname{Hom}_A(P, Q)$ is acyclic for every projective A-module Q. We denote by $\mathbb{K}_{\mathrm{tac}}(\operatorname{Proj} A)$ the full subcategory of totally acyclic complexes in $\mathbb{K}(\operatorname{Proj} A)$, which is a triangulated subcategory; these complexes are also called *complete projective resolutions*. See [AB69], [Chr00, §4.2], [AM02] and [IK06] for properties of these complexes.

We say that an A-module is *Gorenstein projective* if it is isomorphic to $Z^0(P)$ for some totally acyclic complex P of projective A-modules. When A is Gorenstein and has finite Krull dimension, a complex of projective A-modules is totally acyclic if and only if it is acyclic; see [IK06, Corollary 5.5]. Denote by $\operatorname{Gproj}(\underline{\mathbf{Mod}}A)$ the full subcategory of the stable module category $\underline{\mathbf{Mod}}A$ given by the Gorenstein projective objects. In light of the definition of Gorenstein projective modules as syzygies of objects of $\mathbb{K}_{\mathrm{tac}}(\operatorname{Proj} A)$, the following equivalence comes as no surprise.

Proposition 5.10. *Let A be a noetherian ring. There is an equivalence*

$$Z^0(-) : \mathbb{K}_{\mathrm{tac}}(\operatorname{Proj} A) \xrightarrow{\sim} \operatorname{Gproj}(\underline{\mathbf{Mod}}A) \tag{5.9}$$

defined by taking the kernel in degree zero.

Proof. The proof is dual to [Kra05, Proposition 7.2]. □

The relationship between maximal Cohen-Macaulay modules and Gorenstein projective modules is part of the motivation for the subject; see the survey in [EE05b]. For the benefit of the reader who is not familiar with relative homological algebra we include the following statements, most of which can be found in Christensen's excellent book [Chr00].

Let A be a noetherian ring. The G-dimension, or *Gorenstein dimension*, of a finitely generated A-module M is denoted $\operatorname{Gdim}_A(M)$. This dimension was defined by Auslander and Bridger in [AB69]. The A-modules of G-dimension zero form a class $\mathrm{G}(A)$, called the *G-class* of A. If A is local, a finitely generated A-module M is *maximal Cohen-Macaulay* (MCM) if $\operatorname{depth}_A(M) = \dim(A)$. When A is an arbitrary noetherian ring, we say that a finitely generated A-module M is MCM if $M_{\mathfrak{p}}$ is a MCM module over $A_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p} \in \mathit{Spec}(A)$.

Lemma 5.11. *Let A be a Gorenstein ring of finite Krull dimension. The following conditions are equivalent for a finitely generated A-module M:*

(i) M is maximal Cohen-Macaulay.

(ii) M is Gorenstein projective.

(iii) M occurs as a syzygy of an acyclic complex of finitely generated free A-modules.

Proof. By [Chr00, Theorem 4.2.6] the module M is Gorenstein projective precisely when it belongs to $\mathrm{G}(A)$. Membership in the G-class is local; that is, $M \in G(A)$ if and only if $M_{\mathfrak{p}} \in \mathrm{G}(A_{\mathfrak{p}})$ for every prime $\mathfrak{p} \in Spec(A)$ [Chr00, Lemma 1.3.1]. In the local ring $A_{\mathfrak{p}}$ we have the Auslander-Bridger formula [Chr00, Theorem 1.4.8]

$$\mathrm{Gdim}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) + \mathrm{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \dim(A_{\mathfrak{p}}) \tag{5.10}$$

Here we use the fact that $A_{\mathfrak{p}}$ is Gorenstein, so every finitely generated module has finite G-dimension [Chr00, Theorem 1.4.9]. Now $M \in \mathrm{G}(A)$ if and only if $\mathrm{Gdim}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = 0$ for every prime $\mathfrak{p}$, which by (5.10) is equivalent to $\mathrm{depth}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \dim(A_{\mathfrak{p}})$ for every $\mathfrak{p}$. This is the statement that M is maximal Cohen-Macaulay, so we have established $(i) \Leftrightarrow (ii)$. For the equivalence of (ii) and (iii) see [Chr00, Theorem 4.1.4] and [IK06, Corollary 5.5]. □

5.1 The Stable Derived Category of Projective Modules

The material of the previous section specializes to results about the homotopy category $\mathbb{K}(\mathrm{Proj}\, A)$ of projective modules over a commutative ring A. In this section we point out that commutativity is superfluous; the results hold in complete generality.

Setup. In this section A is a noncommutative ring and modules are left A-modules.

The results of this section are straightforward consequences of the results established by Neeman in his papers [Nee06a] and [Nee06c]. We are only able to work in such complete generality (no finiteness conditions on A) by exploiting the theory of well generated triangulated categories, so it is appropriate to say a few words about these categories.

Remark 5.12. In [Nee01b] Neeman introduced the *well generated* triangulated categories as a generalization of the compactly generated ones; see also [Kra01]. Neeman's book shows why this is the correct generality in which to prove theorems like Brown representability.

There are natural examples of triangulated categories that are well generated but not compactly generated. For example, the derived category $\mathbb{D}(\mathcal{A})$ of a Grothendieck abelian category $\mathcal{A}$ is always well generated, but it is not necessarily compactly generated [Nee01a]. Of more relevance to us here, it is known that $\mathbb{K}(\mathrm{Proj}\, A)$ is compactly generated for right coherent rings [Nee06a, Jør05] but Neeman has shown that this is not true in general. The homotopy category $\mathbb{K}(\mathrm{Proj}\, A)$ is, however, always well generated [Nee06a, Theorem 4.8].

Our first task is to prove some technical results characterizing $\mathbb{K}$-projectivity.

Proposition 5.13. *The inclusions*

$$K : \mathbb{K}(\mathrm{Proj}\, A) \longrightarrow \mathbb{K}(A), \quad J : \mathbb{K}(\mathrm{Flat}\, A) \longrightarrow \mathbb{K}(A)$$

both have right adjoints that are exact. That is, the right adjoints send acyclic complexes in $\mathbb{K}(A)$ to acyclic complexes.

Proof. We know from the work of Neeman that $\mathbb{K}(\mathrm{Proj}\, A)$ is well generated, so it satisfies Brown representability [Nee06a, Corollary 4.9]. It follows from [Nee01b, Theorem 8.4.4] that the inclusion K has a right adjoint. The existence of a right adjoint for J is connected with flat covers, and is due to Neeman; see [Nee06c, Theorem 3.3]. It remains to prove that the adjoints are exact. For $M \in \mathbb{K}(A)$ the counit $KK_\rho(M) \longrightarrow M$ fits into a triangle

$$KK_\rho(M) \longrightarrow M \longrightarrow S \longrightarrow \Sigma KK_\rho(M) \tag{5.11}$$

with $S \in \mathbb{K}(\mathrm{Proj}\, A)^\perp$. By Lemma 2.22 we can find a quasi-isomorphism $P \longrightarrow S$ with P a complex of projectives; this morphism must be zero in $\mathbb{K}(A)$, from which we deduce that S is acyclic. From the triangle (5.11) we conclude that the adjoint $K_\rho : \mathbb{K}(A) \longrightarrow \mathbb{K}(\mathrm{Proj}\, A)$ sends acyclic complexes to acyclic complexes, and the same argument applies to J_ρ. $\square$

Recall from Section 2.2 the definition of $\mathbb{K}$-*injective*, $\mathbb{K}$-*projective* and $\mathbb{K}$-*flat* complexes. To be clear, over the noncommutative ring A a complex X of left A-modules is $\mathbb{K}$-*flat* if $Z \otimes_A X$ is acyclic for every acyclic complex Z of right A-modules.

Corollary 5.14. *We have the following classification of orthogonals*

(i) A complex X of projective modules is $\mathbb{K}$-projective if and only if $X \in {}^\perp\, \mathbb{K}_{\mathrm{ac}}(\mathrm{Proj}\, A)$.

(ii) A complex X of flat modules is $\mathbb{K}$-projective if and only if $X \in {}^\perp\, \mathbb{K}_{\mathrm{ac}}(\mathrm{Flat}\, A)$.

(iii) A complex X of flat modules is $\mathbb{K}$-flat if and only if $X \in {}^\perp\, \mathbb{K}_{m,\mathrm{ac}}(\mathrm{Proj}\, A)$.

where the orthogonal in (iii) is taken in $\mathbb{K}_m(\mathrm{Proj}\, A)$. It follows that

(iv) A complex X of projective modules is $\mathbb{K}$-flat if and only if it is $\mathbb{K}$-projective.

Proof. Note that orthogonals are always calculated relative to some ambient triangulated category. In (i) and (ii) above this category is $\mathbb{K}(A)$ and in (iii) it is $\mathbb{K}_m(\mathrm{Proj}\, A)$.

(i) If X is $\mathbb{K}$-projective then it is left orthogonal to *every* acyclic complex, and in particular it is left orthogonal to the acyclic complexes of projective modules, so X belongs to ${}^\perp\, \mathbb{K}_{\mathrm{ac}}(\mathrm{Proj}\, A)$. In the other direction, let X be a complex of projectives left orthogonal to every acyclic complex of projective modules. For an acyclic complex C of modules it follows from Proposition 5.13 that $K_\rho(C)$ is acyclic, so $\mathrm{Hom}_{\mathbb{K}(A)}(X, C) \cong \mathrm{Hom}_{\mathbb{K}(\mathrm{Proj}\, A)}(X, K_\rho(C))$ vanishes, proving that X is $\mathbb{K}$-projective. The claim (ii) is proved similarly.

(iii) The complex X has a $\mathbb{K}$-projective resolution consisting of a $\mathbb{K}$-projective complex P of projective modules and a triangle in $\mathbb{K}(A)$ with C acyclic (see Lemma 2.22)

$$P \longrightarrow X \longrightarrow C \longrightarrow \Sigma P \tag{5.12}$$

Any acyclic complex Z of flat modules is isomorphic, in $\mathbb{K}_m(\mathrm{Proj}\, A)$, to an acyclic complex Z' of projective modules, and using Remark 3.5(iii) we have

$$\mathrm{Hom}_{\mathbb{K}_m(\mathrm{Proj}\, A)}(P, Z) \cong \mathrm{Hom}_{\mathbb{K}(\mathrm{Proj}\, A)}(P, Z') = 0$$

from which we deduce that $P \in {}^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\mathrm{Proj}\, A)$. If X is $\mathbb{K}$-flat then from (5.12) we infer that C is $\mathbb{K}$-flat and thus zero in $\mathbb{K}_m(\mathrm{Proj}\, A)$. Hence $X \cong P$ belongs to ${}^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\mathrm{Proj}\, A)$. In the other direction, if X belongs to this orthogonal then so does C, whence C is orthogonal to itself and therefore zero. We conclude that $X \cong P$ is $\mathbb{K}$-flat ($\mathbb{K}$-flatness is stable under isomorphism in $\mathbb{K}_m(\mathrm{Proj}\, A)$, by the argument of Lemma 5.1).

(iv) The equivalence $\mathbb{K}(\mathrm{Proj}\, A) \xrightarrow{\sim} \mathbb{K}_m(\mathrm{Proj}\, A)$ of Lemma 3.6 identifies $\mathbb{K}_{\mathrm{ac}}(\mathrm{Proj}\, A)$ with $\mathbb{K}_{m,\mathrm{ac}}(\mathrm{Proj}\, A)$ and therefore identifies ${}^{\perp}\mathbb{K}_{\mathrm{ac}}(\mathrm{Proj}\, A)$ with ${}^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\mathrm{Proj}\, A)$. By (i) the first orthogonal consists of the $\mathbb{K}$-projective complexes of projective modules and by (iii) the second orthogonal is the full subcategory of $\mathbb{K}_m(\mathrm{Proj}\, A)$ consisting of the $\mathbb{K}$-flat complexes of flat modules. The claim is now immediate. □

The injective analogue of the next theorem is known for noetherian rings [Kra05] but for general rings it must be more subtle, because it is not even clear *a priori* how to define coproducts in $\mathbb{K}(\mathrm{Inj}\, A)$ for rings without some noetherian hypothesis.

Theorem 5.15. *The sequence*

$$\mathbb{K}_{\mathrm{ac}}(\mathrm{Proj}\, A) \xrightarrow{\mathrm{inc}} \mathbb{K}(\mathrm{Proj}\, A) \xrightarrow{\mathrm{can}} \mathbb{D}(A) \tag{5.13}$$

induces a recollement

$$\mathbb{K}_{\mathrm{ac}}(\mathrm{Proj}\, A) \rightleftarrows \mathbb{K}(\mathrm{Proj}\, A) \rightleftarrows \mathbb{D}(A) \tag{5.14}$$

Proof. The proof is the same as that given for Theorem 5.5, but since many details simplify it is worth giving the full argument here. Firstly, we note that the composite

$$T : {}^{\perp}\mathbb{K}_{\mathrm{ac}}(\mathrm{Proj}\, A) \longrightarrow \mathbb{K}(\mathrm{Proj}\, A) \longrightarrow \mathbb{D}(A)$$

is fully faithful, because by Corollary 5.14(i) the objects of ${}^{\perp}\mathbb{K}_{\mathrm{ac}}(\mathrm{Proj}\, A)$ are precisely the $\mathbb{K}$-projective complexes of projectives. By Lemma 2.22 every complex X in $\mathbb{D}(A)$ admits a $\mathbb{K}$-projective resolution by a complex of projectives, so T is an equivalence. The composite of the quasi-inverse T^{-1} with the inclusion ${}^{\perp}\mathbb{K}_{\mathrm{ac}}(\mathrm{Proj}\, A) \longrightarrow \mathbb{K}(\mathrm{Proj}\, A)$ gives a fully faithful left adjoint for the functor $u : \mathbb{K}(\mathrm{Proj}\, A) \longrightarrow \mathbb{D}(A)$, so it follows from Lemma 2.6 that (5.13) is a colocalization sequence. The triangulated category $\mathbb{K}(\mathrm{Proj}\, A)$ satisfies Brown representability by [Nee06a, Corollary 4.9], so the coproduct preserving functor u admits a right adjoint. We conclude that (5.13) is a recollement, as required. □

Next we give the flat analogue. Note that the recollement above generalizes to schemes, but the recollement below does not.

Corollary 5.16. *The sequence*

$$\mathbb{K}_{\mathrm{ac}}(\mathrm{Flat}\, A) \xrightarrow{\mathrm{inc}} \mathbb{K}(\mathrm{Flat}\, A) \xrightarrow{\mathrm{can}} \mathbb{D}(A) \tag{5.15}$$

induces a recollement

$$\mathbb{K}_{\mathrm{ac}}(\mathrm{Flat}\, A) \rightleftarrows \mathbb{K}(\mathrm{Flat}\, A) \rightleftarrows \mathbb{D}(A) \tag{5.16}$$

Proof. Let $I : \mathbb{K}(\operatorname{Proj} A) \longrightarrow \mathbb{K}(\operatorname{Flat} A)$ denote the inclusion, which has right adjoint I_ρ by Remark 3.5 (this is another result of Neeman). Given a complex F of flat modules, there is a triangle $II_\rho(F) \longrightarrow F \longrightarrow S \longrightarrow \Sigma II_\rho(F)$ with S in $\mathbb{K}(\operatorname{Proj} A)^\perp$. We deduce that S is an acyclic complex, and that the following diagram commutes up to natural equivalence

$$\begin{tikzcd} \mathbb{K}(\operatorname{Flat} A) \arrow[rr, "\mathrm{can}"] \arrow[dr, "I_\rho"'] & & \mathbb{D}(A) \\ & \mathbb{K}(\operatorname{Proj} A) \arrow[ur, "\mathrm{can}"'] & \end{tikzcd}$$

We know from the theorem that $\mathbb{K}(\operatorname{Proj} A) \longrightarrow \mathbb{D}(A)$ has fully faithful left and right adjoints. The same is true of I_ρ by [Nee06c, Remark 3.2]. We conclude that the canonical functor $\mathbb{K}(\operatorname{Flat} A) \longrightarrow \mathbb{D}(A)$ has fully faithful left and right adjoints, and therefore induces a recollement. $\square$

The next result adds to the list of well generated triangulated categories.

Corollary 5.17. *The triangulated category $\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ is well generated. If A is right coherent, then it is compactly generated.*

Proof. We use a characterization of well generated triangulated categories given by Krause [Kra01] since it is more convenient for our purposes. In [Nee06a, Theorem 4.8] Neeman proves that the category $\mathbb{K}(\operatorname{Proj} A)$ is $\aleph_1$-compactly generated: there is a set $\mathcal{S}$ of complexes in $\mathbb{K}(\operatorname{Proj} A)$ satisfying the conditions of [Kra01, Theorem A] for the cardinal $\alpha = \aleph_1$

(G1) An object $X \in \mathbb{K}(\operatorname{Proj} A)$ is zero provided $\operatorname{Hom}_{\mathbb{K}(\operatorname{Proj} A)}(S, X) = 0$ for every $S \in \mathcal{S}$;

(G2) For every set of maps $X_i \longrightarrow Y_i$ in $\mathbb{K}(\operatorname{Proj} A)$ the induced map

$$\operatorname{Hom}_{\mathbb{K}(\operatorname{Proj} A)}(S, \oplus_i X_i) \longrightarrow \operatorname{Hom}_{\mathbb{K}(\operatorname{Proj} A)}(S, \oplus_i Y_i)$$

is surjective for every $S \in \mathcal{S}$ provided $\operatorname{Hom}_{\mathbb{K}(\operatorname{Proj} A)}(S, X_i) \longrightarrow \operatorname{Hom}_{\mathbb{K}(\operatorname{Proj} A)}(S, Y_i)$ is surjective for all i and $S \in \mathcal{S}$.

(G3) The objects of $\mathcal{S}$ are $\aleph_1$-small.

Let $F : \mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A) \longrightarrow \mathbb{K}(\operatorname{Proj} A)$ denote the inclusion, which by Theorem 5.15 admits a left adjoint F_λ. It is straightforward to check that the set $\mathcal{S}' = \{F_\lambda(S) \,|\, S \in \mathcal{S}\}$ satisfies the above conditions (G1)-(G3) for the triangulated category $\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A)$. From [Kra01, Theorem A] we conclude that $\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ is $\aleph_1$-compactly generated and, in particular, is well generated. If A is right coherent then $\mathbb{K}(\operatorname{Proj} A)$ is compactly generated by [Nee06a, Proposition 6.14]. One checks that F_λ sends a compact generating set for $\mathbb{K}(\operatorname{Proj} A)$ to a compact generating set for $\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A)$, so this latter category is compactly generated. $\square$

There is nothing surprising about this corollary. It is essentially the argument given by Iyengar and Krause in [IK06, Theorem 5.3] who prove the result when A is commutative and noetherian of finite Krull dimension, applied to the new fact that $\mathbb{K}(\operatorname{Proj} A)$ is always well generated.

Chapter 6

Closed Monoidal Structure

Many triangulated categories are also closed monoidal categories. The topological example is the homotopy category of spectra with the smash product $- \wedge -$ and function spectra, while the algebraic example is the derived category $\mathbb{D}(A)$ of a (commutative) ring, which has the derived tensor product $- \underset{=}{\otimes}_A -$ and derived Hom, denoted $\mathbb{R}\mathrm{Hom}_A(-,-)$.

It is hardly possible to overstate the importance of these objects. On the algebraic side, the derived structures contain in their cohomology the derived functors Tor and Ext underlying classical homological algebra: for A-modules M, N we have isomorphisms

$$\mathrm{Tor}_i(M,N) \cong H^{-i}(M \underset{=}{\otimes}_A N), \quad \mathrm{Ext}^i(M,N) \cong H^i \mathbb{R}\,\mathrm{Hom}_A(M,N)$$

In this chapter we define the closed monoidal structure on $\mathbb{K}_m(\mathrm{Proj}\, X)$. The tensor product is the ordinary tensor product of complexes, while the function object $\mathbb{R}\mathscr{F}lat(-,-)$ is more exotic. By analogy with the homotopy category of spectra, we call the functor

$$\mathscr{F} \mapsto \mathscr{F}^\circ = \mathbb{R}\mathscr{F}lat(\mathscr{F}, \mathcal{O}_X)$$

the *Spanier-Whitehead dual* on the mock homotopy category of projectives; see [HPS97, Definition A.2.4]. Taking the dual, in this sense, of flat resolutions of bounded complexes of coherent sheaves produces compact objects in $\mathbb{K}_m(\mathrm{Proj}\, X)$, and we will prove in Chapter 7 that *all* the compact objects are of this form.

Another category of interest is the derived category $\mathbb{D}(\mathfrak{Qco}X)$ of quasi-coherent sheaves. Its closed monoidal structure is identical to the structure on the derived category $\mathbb{D}(X)$ of *arbitrary* sheaves of modules, modulo some technical points, discussed in Section 6.1. The relationship between the derived category and $\mathbb{K}_m(\mathrm{Proj}\, X)$ is a recollement (Chapter 5)

$$\mathbb{K}_{m,\mathrm{ac}}(\mathrm{Proj}\, X) \rightleftarrows \mathbb{K}_m(\mathrm{Proj}\, X) \rightleftarrows \mathbb{D}(\mathfrak{Qco}X)$$

When X is regular the mock homotopy category is equivalent to the derived category, and the closed monoidal structure on $\mathbb{K}_m(\mathrm{Proj}\, X)$ reduces to the derived tensor product and derived Hom (Remark 9.8). In general, however, the tensor product and function object in $\mathbb{K}_m(\mathrm{Proj}\, X)$ extend the corresponding structures of $\mathbb{D}(\mathfrak{Qco}X)$ to take into account the acyclic complexes, which carry information about the singularities of X.

We refer the reader to [HPS97, Appendix A] and [MVW06, Definition 8A.1] for general background on triangulated categories that are also closed monoidal categories.

Setup. In this section X is a noetherian scheme and sheaves are defined over X by default.

First we define the tensor product on $\mathbb{K}_m(\operatorname{Proj} X)$. In the homotopy category $\mathbb{K}(\mathfrak{Qco} X)$ of quasi-coherent sheaves we have a tensor product $- \otimes -$ defined for complexes $\mathscr{F}, \mathscr{G}$ of quasi-coherent sheaves by [Lip, (1.5.4)]. In degree n, this complex is

$$(\mathscr{F} \otimes \mathscr{G})^n = \bigoplus_{i+j=n} \mathscr{F}^i \otimes \mathscr{G}^j$$

This tensor product makes $\mathbb{K}(\mathfrak{Qco} X)$ into a *tensor triangulated category*, in the sense of [MVW06, Definition 8A.1]. The corresponding closed structure is discussed in Section 6.1. Because the tensor product of flat sheaves is flat, the tensor product on $\mathbb{K}(\mathfrak{Qco} X)$ restricts to make $\mathbb{K}(\operatorname{Flat} X)$ into a tensor triangulated category.

Lemma 6.1. *If $\mathscr{F}$ is a complex of flat quasi-coherent sheaves and $\mathscr{E}$ belongs to $\mathbb{E}(X)$ then $\mathscr{F} \otimes \mathscr{E}$ also belongs to $\mathbb{E}(X)$.*

Proof. Complexes in $\mathbb{E}(X)$ are characterized by the fact that tensoring with them sends arbitrary complexes to acyclic complexes; see Proposition 3.4. Given a complex $\mathscr{G}$ we have $\mathscr{G} \otimes (\mathscr{F} \otimes \mathscr{E}) \cong (\mathscr{G} \otimes \mathscr{F}) \otimes \mathscr{E}$, which must be acyclic, whence $\mathscr{F} \otimes \mathscr{E} \in \mathbb{E}(X)$. □

The lemma tells us that $\mathbb{E}(X)$ is a tensor ideal, and it follows that the tensor product on $\mathbb{K}(\operatorname{Flat} X)$ descends to a tensor product on the quotient $\mathbb{K}_m(\operatorname{Proj} X) = \mathbb{K}(\operatorname{Flat} X)/\mathbb{E}(X)$, which thus becomes a tensor triangulated category. The precise argument can be found in [MVW06, Proposition 8A.7]. Because $\mathbb{K}_m(\operatorname{Proj} X)$ is compactly generated, we can deduce function objects from Brown representability, as follows.

Proposition 6.2. *The triangulated category $\mathbb{K}_m(\operatorname{Proj} X)$ is a closed symmetric monoidal category, in which the tensor product and function object $\mathbb{R}\mathscr{F}\!lat(-,-)$ are compatible with the triangulation. In particular, there is a natural isomorphism*

$$\operatorname{Hom}_{\mathbb{K}_m(\operatorname{Proj} X)}(\mathscr{F} \otimes \mathscr{G}, \mathscr{H}) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}_m(\operatorname{Proj} X)}(\mathscr{F}, \mathbb{R}\mathscr{F}\!lat(\mathscr{G}, \mathscr{H})) \tag{6.1}$$

Proof. When we say that the structure is *compatible with the triangulation*, we mean it in the precise sense of [HPS97, Definition A.2.1]. We have already shown that $\mathbb{K}_m(\operatorname{Proj} X)$ is a tensor triangulated category with the ordinary tensor product of complexes. For any object $\mathscr{F} \in \mathbb{K}_m(\operatorname{Proj} X)$ this tensor product gives a coproduct preserving functor

$$- \otimes \mathscr{F} : \mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{K}_m(\operatorname{Proj} X)$$

which has, by Brown representability, a right adjoint $\mathbb{R}\mathscr{F}\!lat(\mathscr{F}, -)$. Here we use the fact that X is noetherian to see that $\mathbb{K}_m(\operatorname{Proj} X)$ is compactly generated (Theorem 4.10), so that the Brown representability theorem of Neeman applies (Proposition 2.7).

This defines the function object $\mathbb{R}\mathscr{F}\!lat(-,-)$, but now some work is required to verify that the conditions given in [HPS97, Definition A.2.1] are satisfied. The only problem is that it is not immediately clear that $\mathbb{R}\mathscr{F}\!lat(-,-)$ is a triangulated functor in the first variable. We give a general argument in Appendix C that settles this point for arbitrary tensor triangulated categories satisfying Brown representability and an extra mild axiom on the tensor product, which holds for $\mathbb{K}_m(\operatorname{Proj} X)$ and the other natural examples (see Proposition C.13 and the subsequent remark).

Alternatively, one can prove that $\mathbb{R}\mathscr{F}\!lat(-,-)$ is triangulated in the first variable by giving a more explicit description of this complex, along the same lines that one defines the derived Hom in $\mathbb{D}(X)$. This development is given in Appendix B. Using either approach, we find that the function object is a triangulated functor in both variables. Since the other conditions of [HPS97, Definition A.2.1] are easily checked, the proof is complete. $\square$

The reader may be left cold by this definition. Although the adjunction (6.1) tells us everything we need to know about the function object as an object of $\mathbb{K}_m(\operatorname{Proj} X)$, we have said nothing about it as a *complex*. In practice we will study $\mathbb{R}\mathscr{F}\!lat(-,-)$ by reducing to the local case, where Lemma 6.6 below computes the function object explicitly. In some important special cases, the global function object simplifies; see Corollary 6.13.

Remark 6.3. The reason for the notation $\mathbb{R}\mathscr{F}\!lat(-,-)$ is that there is a closed structure $\mathscr{F}\!lat(-,-)$ on $\mathbb{K}(\operatorname{Flat} X)$, whose derived functor is the function object in $\mathbb{K}_m(\operatorname{Proj} X)$. The closed structure on $\mathbb{K}(\operatorname{Flat} X)$ is not important here, so we relegate it to Appendix B.

We will give a more traditional exposition on the closed monoidal structure of $\mathbb{D}(\mathfrak{Qco} X)$ in Section 6.1, but using Brown representability it is possible to give a shorter proof that bypasses some nonsense about coherators.

Proposition 6.4. *The triangulated category $\mathbb{D}(\mathfrak{Qco} X)$ is closed symmetric monoidal, with tensor product $-\underline{\underline{\otimes}}-$ and function object $\mathbb{R}\mathscr{H}\!om_{\mathrm{qc}}(-,-)$ compatible with the triangulation, and there is a natural isomorphism*

$$\operatorname{Hom}_{\mathbb{D}(\mathfrak{Qco} X)}(\mathscr{F} \underline{\underline{\otimes}} \mathscr{G}, \mathscr{H}) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{D}(\mathfrak{Qco} X)}(\mathscr{F}, \mathbb{R}\mathscr{H}\!om_{\mathrm{qc}}(\mathscr{G}, \mathscr{H}))$$

Proof. In Chapter 5 we identified $\mathbb{D}(\mathfrak{Qco} X)$ with the full subcategory of $\mathbb{K}$-flat complexes in $\mathbb{K}_m(\operatorname{Proj} X)$, by identifying a complex with its resolution. To be precise, we established an equivalence of triangulated categories (see Remark 5.9)

$$^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \xrightarrow{\mathrm{inc}} \mathbb{K}_m(\operatorname{Proj} X) \xrightarrow{\mathrm{can}} \mathbb{D}(\mathfrak{Qco} X) \tag{6.2}$$

where $^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ consists of $\mathbb{K}$-flat complexes of flat quasi-coherent sheaves (Proposition 5.2) and the quasi-inverse of (6.2) sends a complex to its $\mathbb{K}$-flat resolution. Because the tensor product of $\mathbb{K}$-flat complexes is $\mathbb{K}$-flat, $^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ is a tensor triangulated category, and from (6.2) we deduce that $\mathbb{D}(\mathfrak{Qco} X)$ is a tensor triangulated category with the induced structure, which is just the derived tensor product in the usual sense.

Since $\mathbb{D}(\mathfrak{Qco} X)$ is compactly generated by [Nee96, Proposition 2.5] the argument given in Appendix C now applies to prove that $\mathbb{D}(\mathfrak{Qco} X)$ has a function object $\mathbb{R}\mathscr{H}\!om_{\mathrm{qc}}(-,-)$ making it into a closed monoidal category in a way compatible with the triangulation; see Remark C.14. $\square$

The function object $\mathbb{R}\mathscr{H}om_{\mathrm{qc}}(-,-)$ agrees with the usual derived Hom in the derived category $\mathbb{D}(X)$ of arbitrary sheaves of modules, denoted here by $\mathbb{R}\mathscr{H}om(-,-)$, whenever such agreement is possible.

Lemma 6.5. *There is a morphism in $\mathbb{D}(X)$ natural in both variables*

$$\mathbb{R}\mathscr{H}om_{\mathrm{qc}}(\mathscr{F},\mathscr{G}) \longrightarrow \mathbb{R}\mathscr{H}om(\mathscr{F},\mathscr{G})$$

If $\mathbb{R}\mathscr{H}om(\mathscr{F},\mathscr{G})$ has quasi-coherent cohomology, this is an isomorphism in $\mathbb{D}(X)$.

Proof. The canonical functor $\mathbb{D}(\mathfrak{Qco}X) \longrightarrow \mathbb{D}(X)$ is fully faithful and induces an equivalence of $\mathbb{D}(\mathfrak{Qco}X)$ with the subcategory $\mathbb{D}_{\mathrm{qc}}(X) \subseteq \mathbb{D}(X)$ of complexes with quasi-coherent cohomology; see [BN93, Corollary 5.5]. We have a canonical natural isomorphism

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{D}(\mathfrak{Qco}X)}(\mathscr{A},\mathbb{R}\mathscr{H}om_{\mathrm{qc}}(\mathscr{F},\mathscr{G})) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{D}(\mathfrak{Qco}X)}(\mathscr{A} \underset{=}{\otimes} \mathscr{F},\mathscr{G}) && \text{(Adjunction)}\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{D}(X)}(\mathscr{A} \underset{=}{\otimes} \mathscr{F},\mathscr{G}) && \text{(Inclusion)}\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{D}(X)}(\mathscr{A},\mathbb{R}\mathscr{H}om(\mathscr{F},\mathscr{G})) && \text{(Adjunction)}
\end{aligned}$$

The desired morphism corresponds to the identity on the left, when $\mathscr{A}$ is $\mathbb{R}\mathscr{H}om_{\mathrm{qc}}(\mathscr{F},\mathscr{G})$. This proves that $\mathbb{R}\mathscr{H}om_{\mathrm{qc}}(\mathscr{F},\mathscr{G})$ represents $\mathbb{R}\mathscr{H}om(\mathscr{F},\mathscr{G})$ amongst the complexes with quasi-coherent cohomology; if $\mathbb{R}\mathscr{H}om(\mathscr{F},\mathscr{G})$ is already one of these complexes, then the two complexes must be isomorphic in $\mathbb{D}(X)$. $\square$

In the rest of this section we study the function object $\mathbb{R}\mathscr{F}lat(-,-)$. In good situations it is local, and over an open affine it simplifies. Over a ring A we work with modules rather than sheaves, so in place of $\mathbb{K}_m(\mathrm{Proj}\,X)$ we use the equivalent category (Remark 3.5)

$$\mathbb{K}_m(\mathrm{Proj}\,A) = \mathbb{K}(\mathrm{Flat}\,A)/\mathbb{E}(A)$$

The function object on $\mathbb{K}_m(\mathrm{Proj}\,A)$ given by Proposition 6.2 is denoted $\mathbb{R}\mathrm{Flat}(-,-)$ rather than $\mathbb{R}\mathscr{F}lat(-,-)$ to emphasize that it is a complex of modules. In the following lemma we relate this complex to the ordinary Hom complex.

Lemma 6.6. *Let A be a noetherian ring, F a complex of flat A-modules and P a complex of finitely generated projective A-modules. There is an isomorphism in $\mathbb{K}_m(\mathrm{Proj}\,A)$*

$$\mathrm{Hom}_A(P,F) \xrightarrow{\sim} \mathbb{R}\mathrm{Flat}(P,F) \tag{6.3}$$

natural with respect to morphisms of complexes in both variables.

Proof. First, let us make some general comments. Recall that the objects of the orthogonal ${}^{\perp}\mathbb{E}(A)$ in $\mathbb{K}(\mathrm{Flat}\,A)$ are, up to homotopy equivalence, the complexes of projective modules (Remark 3.5(iii)), so given a complex p of projective A-modules and a complex f of flat A-modules, there is an isomorphism (see [Nee01b, Lemma 9.1.5])

$$\mathrm{Hom}_{\mathbb{K}(\mathrm{Flat}\,A)}(p,f) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{K}_m(\mathrm{Proj}\,A)}(p,f) \tag{6.4}$$

The quotient $Q : \mathbb{K}(\mathrm{Flat}\, A) \longrightarrow \mathbb{K}_m(\mathrm{Proj}\, A)$ has a left adjoint Q_λ (Remark 3.5) and given a complex Z of flat A-modules, $Q_\lambda(Z)$ is a complex of projective A-modules isomorphic, in the quotient $\mathbb{K}_m(\mathrm{Proj}\, A)$, to Z.

Now for the proof of the lemma. An infinite product of flat modules over a noetherian ring is flat, so $\mathrm{Hom}_A(P, F)$ is a complex of flat A-modules and it makes sense as an object of $\mathbb{K}_m(\mathrm{Proj}\, A)$. Given a complex Z of flat A-modules, we have an isomorphism

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{K}_m(\mathrm{Proj}\, A)}(Z, \mathrm{Hom}_A(P, F)) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{K}(\mathrm{Flat}\, A)}(Q_\lambda(Z), \mathrm{Hom}_A(P, F)) && \text{(Adjunction)}\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{K}(\mathrm{Flat}\, A)}(Q_\lambda(Z) \otimes P, F) && \text{(Adjunction)}\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{K}_m(\mathrm{Proj}\, A)}(Q_\lambda(Z) \otimes P, F) && \text{(By 6.4)}\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{K}_m(\mathrm{Proj}\, A)}(Z, \mathbb{R}\mathrm{Flat}(P, F)) && \text{(Adjunction)}
\end{aligned}$$

This isomorphism is natural in the variables P, F with respect to morphisms of $\mathbb{K}(\mathrm{Flat}\, A)$, and natural in Z with respect to all morphisms of $\mathbb{K}_m(\mathrm{Proj}\, A)$, so we deduce the desired natural isomorphism (6.3) in $\mathbb{K}_m(\mathrm{Proj}\, A)$. □

In the cases where we care about $\mathbb{R}\mathscr{F}\!lat(\mathscr{F}, \mathscr{G})$ the complex $\mathscr{F}$ will often be something like a complex of vector bundles, which is locally a complex of finitely generated projectives, so the lemma calculates the function object *locally* in many cases of interest. To exploit this knowledge globally, we need to know when there is an isomorphism of the form

$$\mathbb{R}\mathscr{F}\!lat(\mathscr{F}, \mathscr{G})|_U \xrightarrow{\sim} \mathbb{R}\mathscr{F}\!lat(\mathscr{F}|_U, \mathscr{G}|_U)$$

We will prove in Proposition 6.12 that this holds whenever $\mathscr{F}$ is locally a complex of vector bundles and $\mathscr{G}$ is bounded. The first step towards the proof is understanding restriction between affine schemes: that is, we study how $\mathbb{R}\mathrm{Flat}(-,-)$ behaves under extension along a flat ring morphism.

Proposition 6.7. *Given a flat morphism $A \longrightarrow B$ of noetherian rings and complexes F and G of flat A-modules, there is a natural morphism in* $\mathbb{K}_m(\mathrm{Proj}\, B)$

$$\mathbb{R}\mathrm{Flat}_A(F, G) \otimes_A B \longrightarrow \mathbb{R}\mathrm{Flat}_B(F \otimes_A B, G \otimes_A B) \tag{6.5}$$

If each F^i is finitely generated and G is bounded, this is an isomorphism in $\mathbb{K}_m(\mathrm{Proj}\, B)$.

Proof. In any closed monoidal category there are unit and counit morphisms relating the tensor product and function objects. For example, there is a morphism in $\mathbb{K}_m(\mathrm{Proj}\, A)$

$$\mathbb{R}\mathrm{Flat}_A(F, G) \otimes_A F \longrightarrow G$$

Applying $- \otimes_A B$ and rearranging produces a morphism in $\mathbb{K}_m(\mathrm{Proj}\, B)$

$$\{\mathbb{R}\mathrm{Flat}_A(F, G) \otimes_A B\} \otimes_B \{F \otimes_A B\} \cong (\mathbb{R}\mathrm{Flat}_A(F, G) \otimes_A F) \otimes_A B \longrightarrow G \otimes_A B$$

that corresponds under adjunction to a morphism (6.5) in $\mathbb{K}_m(\mathrm{Proj}\, B)$. Suppose that each F^i is a finitely generated projective module and that G is bounded. By Lemma 6.6 we can replace $\mathbb{R}\mathrm{Flat}(-,-)$ by $\mathrm{Hom}(-,-)$ and reduce to checking that the canonical morphism

$$\mathrm{Hom}_A(F, G) \otimes_A B \longrightarrow \mathrm{Hom}_B(F \otimes_A B, G \otimes_A B) \tag{6.6}$$

is an isomorphism in $\mathbb{K}(B)$. This reduction involves checking some compatibility diagrams, but the verification is routine. The right hand side of (6.6) is canonically isomorphic to the complex of B-modules $\operatorname{Hom}_A(F, G \otimes_A B)$ and we are in the situation of the tensor evaluation isomorphism: boundedness of G and finiteness of the objects in F imply the necessary isomorphism

$$\operatorname{Hom}_A(F, G) \otimes_A B \xrightarrow{\sim} \operatorname{Hom}_A(F, G \otimes_A B)$$

which completes the proof. □

Setup. In the rest of this section $U \subseteq X$ is an open subset with inclusion $f : U \longrightarrow X$.

The restriction functor $(-)|_U : \mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{K}_m(\operatorname{Proj} U)$ preserves coproducts and admits by Brown representability a right adjoint (see Proposition 2.7 and Theorem 4.10)

$$\hat{\mathbb{R}} f_* : \mathbb{K}_m(\operatorname{Proj} U) \longrightarrow \mathbb{K}_m(\operatorname{Proj} X)$$

If U is affine this agrees with the ordinary direct image; see Definition 3.9. The adjunction between restriction and direct image can now be "enriched".

Lemma 6.8. *Given complexes of flat quasi-coherent sheaves $\mathscr{G}$ on X and $\mathscr{H}$ on U there is a natural isomorphism in $\mathbb{K}_m(\operatorname{Proj} X)$*

$$\mathbb{R}\mathscr{F}\!lat(\mathscr{G}, \hat{\mathbb{R}} f_* \mathscr{H}) \xrightarrow{\sim} \hat{\mathbb{R}} f_* \mathbb{R}\mathscr{F}\!lat(\mathscr{G}|_U, \mathscr{H})$$

Proof. For a complex $\mathscr{F}$ of flat quasi-coherent sheaves on X we have the following natural isomorphism, with Homs taken in $\mathbb{K}_m(\operatorname{Proj} X)$ or $\mathbb{K}_m(\operatorname{Proj} U)$ as appropriate

$$\begin{aligned}
\operatorname{Hom}(\mathscr{F}, \mathbb{R}\mathscr{F}\!lat(\mathscr{G}, \hat{\mathbb{R}} f_* \mathscr{H})) &\xrightarrow{\sim} \operatorname{Hom}(\mathscr{F} \otimes \mathscr{G}, \hat{\mathbb{R}} f_* \mathscr{H}) \\
&\xrightarrow{\sim} \operatorname{Hom}(\mathscr{F}|_U \otimes \mathscr{G}|_U, \mathscr{H}) \\
&\xrightarrow{\sim} \operatorname{Hom}(\mathscr{F}|_U, \mathbb{R}\mathscr{F}\!lat(\mathscr{G}|_U, \mathscr{H})) \\
&\xrightarrow{\sim} \operatorname{Hom}(\mathscr{F}, \hat{\mathbb{R}} f_* \mathbb{R}\mathscr{F}\!lat(\mathscr{G}|_U, \mathscr{H}))
\end{aligned}$$

from which we infer the desired isomorphism. □

Definition 6.9. A complex $\mathscr{F}$ of flat quasi-coherent sheaves is *locally a complex of vector bundles in* $\mathbb{K}_m(\operatorname{Proj} X)$ if every point $x \in X$ has an open neighborhood $V \subseteq X$ such that $\mathscr{F}|_V$ is isomorphic to a complex of vector bundles in $\mathbb{K}_m(\operatorname{Proj} V)$. It will not cause confusion if we drop the qualifier "*in* $\mathbb{K}_m(\operatorname{Proj} X)$" and we will usually do so.

Let us give two examples that will be of importance later.

Lemma 6.10. *A compact object in $\mathbb{K}_m(\operatorname{Proj} X)$ is locally a complex of vector bundles.*

Proof. Compactness is local by Lemma 3.15, so it suffices to check that for a noetherian ring A every compact object in $\mathbb{K}(\operatorname{Proj} A)$ is isomorphic to a complex of finitely generated projectives. This follows from [Nee06a, Proposition 6.12]. □

Lemma 6.11. *For a bounded above complex $\mathscr{G}$ of coherent sheaves, the $\mathbb{K}$-flat resolution by flat quasi-coherent sheaves of $\mathscr{G}$ is locally a complex of vector bundles.*

Proof. Such a resolution $\mathscr{P} \longrightarrow \mathscr{G}$ exists, and is unique in $\mathbb{K}_m(\operatorname{Proj} X)$, by Remark 5.9. Locally, $\mathscr{G}$ is a bounded above complex of finitely generated modules over a noetherian ring, which admits a resolution by a complex of finitely generated projectives. Uniqueness of $\mathbb{K}$-flat resolutions in the mock homotopy category implies that, locally, $\mathscr{P}$ is isomorphic to such a resolution, which is what we needed to show. □

Finally, the main result.

Proposition 6.12. *Given complexes $\mathscr{F}, \mathscr{G}$ of flat quasi-coherent sheaves on X there is a canonical natural morphism in $\mathbb{K}_m(\operatorname{Proj} U)$*

$$\tau : \mathbb{R}\mathscr{F}\!lat(\mathscr{F}, \mathscr{G})|_U \longrightarrow \mathbb{R}\mathscr{F}\!lat(\mathscr{F}|_U, \mathscr{G}|_U)$$

When $\mathscr{F}$ is locally a complex of vector bundles and $\mathscr{G}$ is bounded, this is an isomorphism in $\mathbb{K}_m(\operatorname{Proj} U)$.

Proof. Using the unit of adjunction $\mathscr{G} \longrightarrow \hat{\mathbb{R}} f_*(\mathscr{G}|_U)$ and Lemma 6.8, we have a canonical natural morphism in $\mathbb{K}_m(\operatorname{Proj} X)$

$$\mathbb{R}\mathscr{F}\!lat(\mathscr{F}, \mathscr{G}) \longrightarrow \mathbb{R}\mathscr{F}\!lat(\mathscr{F}, \hat{\mathbb{R}} f_*(\mathscr{G}|_U)) \xrightarrow{\sim} \hat{\mathbb{R}} f_* \mathbb{R}\mathscr{F}\!lat(\mathscr{F}|_U, \mathscr{G}|_U)$$

corresponding under adjunction to the morphism τ. Assume that $\mathscr{F}$ is locally a complex of vector bundles and that $\mathscr{G}$ is bounded. To prove that τ is an isomorphism we use two reductions: firstly to the case of affine open U, and secondly to the case where both X and U are affine. For the first reduction, note that for an open affine $W \subseteq U$ we have a commutative diagram in $\mathbb{K}_m(\operatorname{Proj} W)$

$$\begin{tikzcd}
\mathbb{R}\mathscr{F}\!lat(\mathscr{F}, \mathscr{G})|_U|_W \arrow[rr, "\tau|_W"] \arrow[dr, bend right, "\tau''"'] & & \mathbb{R}\mathscr{F}\!lat(\mathscr{F}|_U, \mathscr{G}|_U)|_W \arrow[dl, bend left, "\tau'"] \\
& \mathbb{R}\mathscr{F}\!lat(\mathscr{F}|_W, \mathscr{G}|_W) &
\end{tikzcd}$$

Suppose that the proposition holds when U is affine. Then τ', τ'' are both isomorphisms in $\mathbb{K}_m(\operatorname{Proj} W)$, and we conclude that $\tau|_W$ is also an isomorphism. Since W was an arbitrary affine open subset, τ is an isomorphism and the proof is complete.

Thus, we have reduced to the case where U is affine. With $\mathscr{F}$ fixed, we denote by $\mathcal{L}$ the triangulated subcategory of $\mathbb{K}_m(\operatorname{Proj} X)$ consisting of complexes $\mathscr{G}$ that make τ an isomorphism in $\mathbb{K}_m(\operatorname{Proj} X)$. To show that a particular bounded complex $\mathscr{G}$ belongs to this subcategory it suffices, by Corollary 3.14, to show that $g_*(\mathscr{G}|_V)$ belongs to $\mathcal{L}$ for every affine open subset $V \subseteq X$ with inclusion $g : V \longrightarrow X$. Writing $h : U \cap V \longrightarrow U$ for the inclusion, we have

$$\mathbb{R}\mathscr{F}\!lat(\mathscr{F}, g_*(\mathscr{G}|_V))|_U \cong \{g_* \mathbb{R}\mathscr{F}\!lat(\mathscr{F}|_V, \mathscr{G}|_V)\}|_U \cong h_* \{\mathbb{R}\mathscr{F}\!lat(\mathscr{F}|_V, \mathscr{G}|_V)|_{U\cap V}\}$$
$$\mathbb{R}\mathscr{F}\!lat(\mathscr{F}|_U, g_*(\mathscr{G}|_V)|_U) \cong \mathbb{R}\mathscr{F}\!lat(\mathscr{F}|_U, h_*(\mathscr{G}|_{U\cap V})) \cong h_* \mathbb{R}\mathscr{F}\!lat(\mathscr{F}|_{U\cap V}, \mathscr{G}|_{U\cap V})$$

We conclude that it is enough to establish the isomorphism

$$\mathbb{R}\mathscr{F}\!lat(\mathscr{F}|_V, \mathscr{G}|_V)|_{U\cap V} \xrightarrow{\sim} \mathbb{R}\mathscr{F}\!lat(\mathscr{F}|_{U\cap V}, \mathscr{G}|_{U\cap V}) \tag{6.7}$$

so we reduce to the case where both X and U are affine. In fact, examining the statement of Corollary 3.14, it suffices in (6.7) to consider open affines V that are finite intersections of elements in an affine open cover of X, so we can reduce to the case where X and U are both affine *and* $\mathscr{F}$ is a complex of vector bundles on X. In this case, Proposition 6.7 applies to complete the proof. □

Corollary 6.13. *For a complex $\mathscr{V}$ of vector bundles there is an isomorphism*

$$\mathscr{H}om(\mathscr{V}, \mathcal{O}_X) \xrightarrow{\sim} \mathbb{R}\mathscr{F}\!lat(\mathscr{V}, \mathcal{O}_X) \tag{6.8}$$

in the category $\mathbb{K}_m(\operatorname{Proj} X)$

Proof. The morphism $\mathscr{H}om(\mathscr{V}, \mathcal{O}_X) \otimes \mathscr{V} \longrightarrow \mathcal{O}_X$ corresponds, via the adjunction between the tensor product and $\mathbb{R}\mathscr{F}\!lat(-,-)$, to a morphism (6.8) that we claim is an isomorphism. By Proposition 6.12 this is a local question. Over an affine scheme we must show that for a complex P of finitely generated projectives the canonical morphism

$$\operatorname{Hom}_A(P, A) \longrightarrow \mathbb{R}\mathrm{Flat}(P, A)$$

is an isomorphism in $\mathbb{K}_m(\operatorname{Proj} A)$, which is a consequence of Lemma 6.6. □

6.1 The Derived Category of Quasi-coherent Sheaves

A significant role is played in modern algebraic geometry by the derived category $\mathbb{D}(X)$ of sheaves of modules over a scheme X. We are really only interested in the quasi-coherent sheaves, so one often restricts to the subcategory $\mathbb{D}_{\mathrm{qc}}(X) \subseteq \mathbb{D}(X)$ of complexes with quasi-coherent cohomology. This has the advantage of being compactly generated. By [BN93, Corollary 5.5] there is an equivalence of triangulated categories

$$\mathbb{D}(\mathfrak{Qco} X) \xrightarrow{\sim} \mathbb{D}_{\mathrm{qc}}(X) \tag{6.9}$$

It would be cleaner to work entirely in $\mathbb{D}(\mathfrak{Qco} X)$, but unfortunately our constructions may produce sheaves that are not quasi-coherent. A common example is the sheaf $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ for quasi-coherent sheaves $\mathscr{F}$ and $\mathscr{G}$, which need not be quasi-coherent unless there is some finiteness hypothesis on $\mathscr{F}$.

The solution is to replace $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ by a sheaf $\mathscr{H}om_{\mathrm{qc}}(\mathscr{F}, \mathscr{G})$ that is quasi-coherent for arbitrary quasi-coherent sheaves $\mathscr{F}, \mathscr{G}$, and which agrees with $\mathscr{H}om(\mathscr{F}, \mathscr{G})$ whenever this sheaf happens to be quasi-coherent. This much is well-known, but less well-known is the definition of the derived tensor product and Hom in $\mathbb{D}(\mathfrak{Qco} X)$ that does not appeal to (6.9). We give two approaches: one short, and one long. The short definition via Brown representability has already been given in Proposition 6.4. The long definition is the more traditional one via derived functors, given at the end of this section. The main focus of this section is the definition of the function object in the homotopy category $\mathbb{K}(\mathfrak{Qco}X)$, which we will need in Chapter 8. The subsequent "long" definition of the function object in the derived category $\mathbb{D}(\mathfrak{Qco}X)$ is optional and the reader can safely skip it. There is an independent treatment of the closed monoidal structure on $\mathbb{D}(\mathfrak{Qco}X)$ in a recent paper of Alonso, Jeremías, Pérez and Vale [AJPV07] who go on to show that $\mathbb{D}(\mathfrak{Qco}X)$ is a stable homotopy category. See also the discussion following Corollary 2.5 in [Hov01].

Setup. Throughout this section X is a fixed scheme and sheaves are defined over X.

In the homotopy category $\mathbb{K}(X)$ of sheaves of modules we have internal structures $-\otimes-$ and $\mathscr{H}om(-,-)$ defined for complexes $\mathscr{F},\mathscr{G}$ of sheaves of modules by [Lip, (1.5.3),(1.5.4)]. In degree n, these complexes are defined by

$$(\mathscr{F}\otimes\mathscr{G})^n = \bigoplus_{i+j=n}\mathscr{F}^i\otimes\mathscr{G}^j$$
$$\mathscr{H}om^n(\mathscr{F},\mathscr{G}) = \prod_{q\in\mathbb{Z}}\mathscr{H}om(\mathscr{F}^q,\mathscr{G}^{q+n})$$

Given a quasi-coherent sheaf $\mathscr{F}$ the functor $\mathscr{H}om(\mathscr{F},-)$ may not preserve quasi-coherence. Nonetheless, the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves *must* possess an internal Hom, because the functor $-\otimes\mathscr{F}:\mathfrak{Qco}(X)\longrightarrow\mathfrak{Qco}(X)$ preserves colimits, and therefore has a right adjoint $\mathscr{H}om_{\mathrm{qc}}(\mathscr{F},-)$. Over an affine scheme $X = Spec(A)$ the uniqueness of such an adjoint implies that for A-modules M,N we have an isomorphism in $\mathfrak{Qco}(X)$

$$\mathscr{H}om_{\mathrm{qc}}\Big(\widetilde{M},\widetilde{N}\Big)\cong\mathrm{Hom}_A(M,N)^{\sim}$$

The sheaf $\mathscr{H}om_{\mathrm{qc}}(\mathscr{F},\mathscr{G})$ can be calculated by a Čech argument [TT90, Appendix B.14] and it must agree with $\mathscr{H}om(\mathscr{F},\mathscr{G})$ when $\mathscr{F}$ is locally finitely presented (e.g. coherent on a noetherian scheme). We find it convenient to define $\mathscr{H}om_{\mathrm{qc}}(-,-)$ using the *coherator*. Recall that $\mathfrak{Mod}(X)$ denotes the category of arbitrary sheaves of modules on X.

Lemma 6.14. *The inclusion $\mathfrak{Qco}(X)\longrightarrow\mathfrak{Mod}(X)$ has a right adjoint C, that we call the* coherator.

Proof. By our blanket hypothesis the scheme X is quasi-compact and separated, so $\mathfrak{Qco}(X)$ is Grothendieck abelian and the inclusion has a right adjoint by the Special Adjoint Functor Theorem. □

For the next pair of results we fix the coherator C for our scheme X.

Proposition 6.15. *The category $\mathfrak{Qco}(X)$ is a closed symmetric monoidal category with the usual tensor product and the function object $\mathscr{H}om_{\mathrm{qc}}(-,-)$ defined by*

$$\mathscr{H}om_{\mathrm{qc}}(\mathscr{F},\mathscr{G}) = C\mathscr{H}om(\mathscr{F},\mathscr{G})$$

In particular, there is a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{Qco}(X)}(\mathscr{F}\otimes\mathscr{G},\mathscr{H})\cong\mathrm{Hom}_{\mathfrak{Qco}(X)}(\mathscr{F},\mathscr{H}om_{\mathrm{qc}}(\mathscr{G},\mathscr{H}))$$

The internal structures in $\mathfrak{Qco}(X)$ pass to complexes in such a way as to make the triangulated category $\mathbb{K}(\mathfrak{Qco}X)$ into a closed symmetric monoidal category. There is one subtle point involving products in $\mathfrak{Qco}(X)$, which may not agree with products in $\mathfrak{Mod}(X)$.

Remark 6.16. Because $\mathfrak{Qco}(X)$ is Grothendieck abelian it has products, denoted here by $\prod^{qc}$. These are calculated by applying the coherator to products in $\mathfrak{Mod}(X)$. Note that the product in $\mathfrak{Qco}(X)$ is *not* exact, and therefore not local; see [Kra05, Example 4.9].

Proposition 6.17. *The triangulated category $\mathbb{K}(\mathfrak{Qco}X)$ is closed symmetric monoidal, with tensor product and function object $\mathscr{H}om_{\mathrm{qc}}(-,-)$ compatible with the triangulation, and in degree n we have*

$$\mathscr{H}om^n_{\mathrm{qc}}(\mathscr{F},\mathscr{G}) = C\mathscr{H}om^n(\mathscr{F},\mathscr{G}) = \prod_{q\in\mathbb{Z}}^{\mathrm{qc}} \mathscr{H}om_{\mathrm{qc}}(\mathscr{F}^q,\mathscr{G}^{q+n}) \tag{6.10}$$

In particular, there is a natural isomorphism

$$\mathrm{Hom}_{\mathbb{K}(\mathfrak{Qco}X)}(\mathscr{F}\otimes\mathscr{G},\mathscr{H}) \cong \mathrm{Hom}_{\mathbb{K}(\mathfrak{Qco}X)}(\mathscr{F},\mathscr{H}om_{\mathrm{qc}}(\mathscr{G},\mathscr{H}))$$

Proof. The function object $\mathscr{H}om(-,-)$ in $\mathbb{K}(X)$ becomes, after applying the coherator, a function object $\mathscr{H}om_{\mathrm{qc}}(-,-) = C\mathscr{H}om(-,-)$ for $\mathbb{K}(\mathfrak{Qco}X)$. Together with the usual tensor product this makes $\mathbb{K}(\mathfrak{Qco}X)$ a closed symmetric monoidal category, and the structure is clearly compatible with the triangulation. We deduce (6.10) from the fact that C has a left adjoint, so it sends products to products. □

In the rest of this section we give the "long" definition of the closed monoidal structure on the derived category $\mathbb{D}(\mathfrak{Qco}X)$ of quasi-coherent sheaves. This will not be used in the sequel and is included only for completeness. To make $\mathbb{D}(\mathfrak{Qco}X)$ into a closed monoidal category we derive the tensor and Hom in $\mathbb{K}(\mathfrak{Qco}X)$. The definitions are standard: replace one variable in $-\otimes-$ with a $\mathbb{K}$-flat resolution and the second variable in $\mathscr{H}om_{\mathrm{qc}}(-,-)$ with a $\mathbb{K}$-injective resolution, working throughout with quasi-coherent sheaves and keeping in mind the distinction between $\mathbb{K}$-injective complexes in $\mathfrak{Qco}(X)$ and $\mathfrak{Mod}(X)$. Despite this simple description there is some effort involved in getting the details right; one way to avoid some of these details is using adjoints: because $\mathfrak{Qco}(X)$ is Grothendieck abelian the quotient $q : \mathbb{K}(\mathfrak{Qco}X) \longrightarrow \mathbb{D}(\mathfrak{Qco}X)$ induces a localization sequence (see Remark 2.20)

$$\begin{tikzcd} \mathbb{K}_{\mathrm{ac}}(\mathfrak{Qco}X) \arrow[r, shift left] & \mathbb{K}(\mathfrak{Qco}X) \arrow[l, shift left] \arrow[r, shift left, "q"] & \mathbb{D}(\mathfrak{Qco}X) \arrow[l, shift left, "q_\rho"] \end{tikzcd}$$

where the right adjoint q_ρ sends a complex to its $\mathbb{K}$-injective resolution. This is well-known, but new to this thesis is the construction of $\mathbb{K}$-flat resolutions by an adjoint functor. By Theorem 5.5 the canonical functor $U : \mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{D}(\mathfrak{Qco}X)$ induces a recollement

$$\begin{tikzcd} \mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \arrow[r] & \mathbb{K}_m(\operatorname{Proj} X) \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r] & \mathbb{D}(\mathfrak{Qco}X) \arrow[l, shift right=2, "U_\lambda"'] \arrow[l, shift left=2, "U_\rho"] \end{tikzcd}$$

The left adjoint U_λ sends a complex to a $\mathbb{K}$-flat resolution by flat quasi-coherent sheaves, and does so functorially; see Remark 5.9. We will use these adjoints to define the tensor product and function objects in $\mathbb{D}(\mathfrak{Qco}X)$, but first we need to prove a technical lemma.

Lemma 6.18. *Given complexes $\mathscr{F}$ and $\mathscr{I}$ of quasi-coherent sheaves that are, respectively, $\mathbb{K}$-flat and $\mathbb{K}$-injective, the complex $\mathscr{H}om_{\mathrm{qc}}(\mathscr{F}, \mathscr{I})$ is $\mathbb{K}$-injective.*

Proof. When we say that a complex of quasi-coherent sheaves is $\mathbb{K}$-injective we mean that it belongs to the orthogonal $\mathbb{K}_{\mathrm{ac}}(\mathfrak{Qco}X)^{\perp}$ in $\mathbb{K}(\mathfrak{Qco}X)$. Given $\mathscr{C} \in \mathbb{K}_{\mathrm{ac}}(\mathfrak{Qco}X)$ we have

$$\mathrm{Hom}_{\mathbb{K}(\mathfrak{Qco}X)}(\mathscr{C}, \mathscr{H}om_{\mathrm{qc}}(\mathscr{F}, \mathscr{I})) \cong \mathrm{Hom}_{\mathbb{K}(\mathfrak{Qco}X)}(\mathscr{C} \otimes \mathscr{F}, \mathscr{I})$$

which is zero, because $\mathscr{C} \otimes \mathscr{F}$ is acyclic. This proves the claim. $\square$

Proposition 6.19. *The triangulated category $\mathbb{D}(\mathfrak{Qco}X)$ is a closed symmetric monoidal category, with tensor product $- \underline{\underline{\otimes}} -$ and function object $\mathbb{R}\mathscr{H}om_{\mathrm{qc}}(-,-)$ compatible with the triangulation. These complexes are defined by*

$$\mathscr{F} \underline{\underline{\otimes}} \mathscr{G} = \mathscr{F} \otimes U_{\lambda}(\mathscr{G})$$
$$\mathbb{R}\mathscr{H}om_{\mathrm{qc}}(\mathscr{F}, \mathscr{G}) = \mathscr{H}om_{\mathrm{qc}}(\mathscr{F}, q_{\rho}(\mathscr{G}))$$

In particular, there is a natural isomorphism

$$\mathrm{Hom}_{\mathbb{D}(\mathfrak{Qco}X)}(\mathscr{F} \underline{\underline{\otimes}} \mathscr{G}, \mathscr{H}) \cong \mathrm{Hom}_{\mathbb{D}(\mathfrak{Qco}X)}(\mathscr{F}, \mathbb{R}\mathscr{H}om_{\mathrm{qc}}(\mathscr{G}, \mathscr{H}))$$

Proof. The tensor product and function object, as given, are well-defined on objects. Let us check that these definitions make sense for morphisms in the derived category, beginning with the tensor product $- \underline{\underline{\otimes}} -$. Fix a complex $\mathscr{F} \in \mathbb{D}(\mathfrak{Qco}X)$ and consider the composite

$$\mathbb{K}(\mathrm{Flat}\, X) \xrightarrow{\mathrm{inc}} \mathbb{K}(\mathfrak{Qco}X) \xrightarrow{\mathscr{F} \otimes -} \mathbb{K}(\mathfrak{Qco}X) \xrightarrow{\mathrm{can}} \mathbb{D}(\mathfrak{Qco}X) \tag{6.11}$$

For $\mathscr{E}$ in $\mathbb{E}(X)$ the complex $\mathscr{F} \otimes \mathscr{E}$ is acyclic, by Proposition 3.4(*iv*). Hence the composite (6.11) vanishes on $\mathbb{E}(X)$, so it induces a triangulated functor out of the Verdier quotient $\mathbb{K}_m(\mathrm{Proj}\, X)$. Composing with the left adjoint U_{λ}, which takes a $\mathbb{K}$-flat resolution, we have the derived tensor product with $\mathscr{F}$

$$\mathscr{F} \underline{\underline{\otimes}} - : \ \mathbb{D}(\mathfrak{Qco}X) \xrightarrow{U_{\lambda}} \mathbb{K}_m(\mathrm{Proj}\, X) \xrightarrow{\mathscr{F} \otimes -} \mathbb{D}(\mathfrak{Qco}X)$$

Functoriality in the first variable is handled similarly, so we have defined a bifunctor $- \underline{\underline{\otimes}} -$ triangulated in each variable. The functoriality of $\mathbb{R}\mathscr{H}om_{\mathrm{qc}}(-,-)$ is identical and left to the reader. Observe that we use a subscript "qc" for the function object but not the tensor product; this is because our $\mathbb{K}$-flat resolution $U_{\lambda}(\mathscr{G})$ is of the same type used to define the derived tensor product in $\mathbb{D}(X)$, so the two tensor products agree up to canonical isomorphism and there is no need to distinguish them in the notation. It remains to give the adjunction between the tensor product and function objects. Given complexes $\mathscr{F}, \mathscr{G}$ and $\mathscr{H}$ of quasi-coherent sheaves we have

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{D}(\mathfrak{Qco}X)}(\mathscr{F} \underline{\underline{\otimes}} \mathscr{G}, \mathscr{H}) &= \mathrm{Hom}_{\mathbb{D}(\mathfrak{Qco}X)}(\mathscr{F} \otimes U_{\lambda}(\mathscr{G}), \mathscr{H}) && \text{(Definition)}\\
&\cong \mathrm{Hom}_{\mathbb{K}(\mathfrak{Qco}X)}(\mathscr{F} \otimes U_{\lambda}(\mathscr{G}), q_{\rho}(\mathscr{H})) && \text{(Adjunction)}\\
&\cong \mathrm{Hom}_{\mathbb{K}(\mathfrak{Qco}X)}(\mathscr{F}, \mathscr{H}om_{\mathrm{qc}}(U_{\lambda}(\mathscr{G}), q_{\rho}(\mathscr{H}))) && \text{(Adjunction)}\\
&\cong \mathrm{Hom}_{\mathbb{D}(\mathfrak{Qco}X)}(\mathscr{F}, \mathscr{H}om_{\mathrm{qc}}(\mathscr{G}, q_{\rho}(\mathscr{H}))) && \text{(Lemma 6.18)}\\
&= \mathrm{Hom}_{\mathbb{D}(\mathfrak{Qco}X)}(\mathscr{F}, \mathbb{R}\mathscr{H}om_{\mathrm{qc}}(\mathscr{G}, \mathscr{H})) && \text{(Definition)}
\end{aligned}$$

The coherence diagrams that make $\mathbb{D}(\mathfrak{Qco}X)$ closed symmetric monoidal are verified to be commutative in the same way that one checks the analogous properties for $\mathbb{D}(X)$, and similarly one checks compatibility with the triangulation. $\square$

Remark 6.20. The two function objects $\mathbb{R}\mathscr{H}om_{\mathrm{qc}}(-,-)$ defined in Proposition 6.4 and Proposition 6.19 must agree, because they are uniquely determined by the adjunction with the tensor product which is the same in both cases.

Chapter 7

Classifying the Compact Objects

Let X be a noetherian scheme. We have defined the mock homotopy category $\mathbb{K}_m(\operatorname{Proj} X)$ of projectives and its subcategory $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ of acyclic complexes, and shown that both are compactly generated triangulated categories (see Theorem 4.10 and Theorem 5.5). In this chapter we study the compact objects in these categories, and achieve a classification in terms of an equivalence (Theorem 7.4)

$$\mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)^{\mathrm{op}} \xrightarrow{\sim} \mathbb{K}^c_m(\operatorname{Proj} X) \tag{7.1}$$

and an equivalence up to direct factors (Theorem 7.9)

$$(\mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)/\operatorname{Perf}(X))^{\mathrm{op}} \xrightarrow{\sim} \mathbb{K}^c_{m,\mathrm{ac}}(\operatorname{Proj} X) \tag{7.2}$$

where $\mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)$ denotes the bounded derived category of coherent sheaves, and $\operatorname{Perf}(X)$ is the full subcategory of perfect complexes. This was already known for affine schemes; see [Jør05, Theorem 3.2] and [IK06, Theorem 5.3]. Our approach is different, because the lack of projective sheaves dictates that we work throughout with flat resolutions.

Setup. In this section X is a noetherian scheme and sheaves are defined over X by default.

The equivalence (7.1) sends a coherent sheaf to a complex of flat quasi-coherent sheaves which is compact in $\mathbb{K}_m(\operatorname{Proj} X)$. Let us describe briefly how this identification works. A result of Krause identifies compact objects in the homotopy category $\mathbb{K}(\operatorname{Inj} X)$ of injective quasi-coherent sheaves with bounded complexes of coherent sheaves, via an equivalence of triangulated categories [Kra05, Proposition 2.3]

$$\mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X) \xrightarrow{\sim} \mathbb{K}^c(\operatorname{Inj} X)$$

which identifies a coherent sheaf $\mathscr{G}$ with its injective resolution. The projective case must be more subtle, because the flat resolution of $\mathscr{G}$ is not necessarily compact in $\mathbb{K}_m(\operatorname{Proj} X)$ (if it were, then $\mathscr{G}$ would be a perfect complex). Instead, the *Spanier-Whitehead dual* of the flat resolution is compact, which explains the contravariance in (7.1). This dual was introduced in Chapter 6 and denoted there by

$$(-)^\circ = \mathbb{R}\mathscr{F}\!\mathit{lat}(-, \mathcal{O}_X)$$

More generally, the compact object in $\mathbb{K}_m(\operatorname{Proj} X)$ corresponding to a bounded complex of coherent sheaves is the Spanier-Whitehead dual of its $\mathbb{K}$-flat resolution. To deal effectively with $\mathbb{K}$-flat resolutions, we use the results of Chapter 5. Recall that the canonical functor $U : \mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{D}(\mathfrak{Qco} X)$ determines a recollement (Theorem 5.5)

$$\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \rightleftarrows \mathbb{K}_m(\operatorname{Proj} X) \underset{U_\rho}{\overset{U_\lambda}{\rightleftarrows}} \mathbb{D}(\mathfrak{Qco} X) \tag{7.3}$$

where the left adjoint U_λ of U sends a complex in $\mathbb{D}(\mathfrak{Qco} X)$ to its $\mathbb{K}$-flat resolution by flat quasi-coherent sheaves (Remark 5.9). Before continuing, let us clear up a technical point.

Remark 7.1. One advantage of noetherian schemes is that we are never confused about the meaning of the term “bounded derived category of coherent sheaves”, for which there are three candidates

(i) $\mathbb{D}^b(\mathfrak{Coh} X)$: the bounded derived category of the category $\mathfrak{Coh}(X)$ of coherent sheaves.

(ii) $\mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco} X)$: the subcategory of complexes with bounded coherent cohomology in the derived category $\mathbb{D}(\mathfrak{Qco} X)$ of quasi-coherent sheaves.

(iii) $\mathbb{D}^b_{\mathrm{coh}}(X)$: the subcategory of complexes with bounded coherent cohomology in the derived category $\mathbb{D}(X)$ of sheaves of modules.

The inclusions $\mathfrak{Coh}(X) \longrightarrow \mathfrak{Qco}(X)$ and $\mathfrak{Qco}(X) \longrightarrow \mathfrak{Mod}(X)$ yield equivalences of triangulated categories, by [Ver96, Proposition III.2.4.1] and [BN93, Corollary 5.5] respectively

$$\mathbb{D}^b(\mathfrak{Coh} X) \xrightarrow{\sim} \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco} X) \xrightarrow{\sim} \mathbb{D}^b_{\mathrm{coh}}(X) \tag{7.4}$$

Due to our preference for quasi-coherent sheaves, the *bounded derived category of coherent sheaves* hereafter means $\mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco} X)$. It is useful to know that every object of this category is, up to isomorphism, a bounded complex of coherent sheaves. This consequence of (7.4) will be used often, and without explicit mention.

In the first proposition we show that taking the Spanier-Whitehead dual of a $\mathbb{K}$-flat resolution is right adjoint to sending a complex of flat quasi-coherent sheaves to its Spanier-Whitehead dual, considered as an object of $\mathbb{D}(\mathfrak{Qco} X)^{\mathrm{op}}$.

Proposition 7.2. *There is an adjoint pair of triangulated functors*

$$\mathbb{K}_m(\operatorname{Proj} X) \underset{(-)^\circ U_\lambda}{\overset{U(-)^\circ}{\rightleftarrows}} \mathbb{D}(\mathfrak{Qco} X)^{op}$$

where $U(-)^\circ$ is left adjoint to $(-)^\circ U_\lambda$.

Proof. Let a complex $\mathscr{A}$ of quasi-coherent sheaves and a complex $\mathscr{F}$ of flat quasi-coherent sheaves be given. Writing “$\operatorname{Hom}_{\mathbb{K}}(-,-)$” for $\operatorname{Hom}_{\mathbb{K}_m(\operatorname{Proj} X)}(-,-)$ and “$\operatorname{Hom}_{\mathbb{D}}(-,-)$” for

$\mathrm{Hom}_{\mathbb{D}(\mathfrak{Qco}X)}(-,-)$, there is a natural isomorphism

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{D}}(\mathscr{A}, U(\mathscr{F}^{\circ})) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{K}}(U_\lambda(\mathscr{A}), \mathbb{R}\mathscr{F}\!lat(\mathscr{F}, \mathcal{O}_X)) && \text{(Adjunction)}\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{K}}(U_\lambda(\mathscr{A}) \otimes \mathscr{F}, \mathcal{O}_X) && \text{(Adjunction)}\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{K}}(\mathscr{F} \otimes U_\lambda(\mathscr{A}), \mathcal{O}_X) && \text{(Symmetry)}\\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{K}}(\mathscr{F}, (U_\lambda(\mathscr{A}))^{\circ}) && \text{(Adjunction)}
\end{aligned}$$

which establishes the desired adjunction. □

We will show that $U(-)^{\circ}$ restricts to an equivalence of the subcategory $\mathbb{K}_m^c(\mathrm{Proj}\,X)$ of compact objects with the subcategory $\mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)^{\mathrm{op}}$ of complexes with bounded coherent cohomology. But first we need to make some small observations.

Lemma 7.3. *If $\mathscr{F}$ is a complex of flat quasi-coherent sheaves that is locally a complex of vector bundles in $\mathbb{K}_m(\mathrm{Proj}\,X)$ then $\mathscr{F}^{\circ}$ also has this property, and the canonical morphism*

$$\mathscr{F} \longrightarrow \mathscr{F}^{\circ\circ}$$

is an isomorphism in $\mathbb{K}_m(\mathrm{Proj}\,X)$. If $\mathscr{F}$ is compact in $\mathbb{K}_m(\mathrm{Proj}\,X)$ then $\mathscr{F}^{\circ}$ is $\mathbb{K}$-flat.

Proof. By Proposition 6.12 these are all local questions, so we can reduce to proving the following statements for a noetherian ring A (using the notation of Chapter 6):

(a) If P is a complex of finitely generated projective A-modules, then $\mathbb{R}\mathrm{Flat}(P, A)$ is a complex of finitely generated projectives (up to isomorphism) and

$$P \longrightarrow \mathbb{R}\mathrm{Flat}(\mathbb{R}\mathrm{Flat}(P, A), A)$$

is an isomorphism in $\mathbb{K}_m(\mathrm{Proj}\,A)$.

(b) Any compact object P in $\mathbb{K}_m(\mathrm{Proj}\,A)$ has $\mathbb{K}$-flat dual $\mathbb{R}\mathrm{Flat}(P, A)$.

In (a) we can, by Lemma 6.6, replace $\mathbb{R}\mathrm{Flat}(P, A)$ by the isomorphic complex $\mathrm{Hom}_A(P, A)$ in which case the claims are obvious. (b) We know from [Nee06a, Proposition 6.12] that any compact object P in $\mathbb{K}(\mathrm{Proj}\,A)$ is, up to homotopy equivalence, bounded below. Hence $\mathrm{Hom}_A(P, A) \cong \mathbb{R}\mathrm{Flat}(P, A)$ is, up to homotopy equivalence, a bounded above complex of projective modules, therefore $\mathbb{K}$-flat. □

Theorem 7.4. *The functor $U(-)^{\circ}$ restricts to an equivalence*

$$U(-)^{\circ} : \mathbb{K}_m^c(\mathrm{Proj}\,X) \xrightarrow{\sim} \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)^{op}$$

with quasi-inverse $(-)^{\circ}U_\lambda$.

Proof. The proof has two parts. In part (A) we will prove that the functors of Proposition 7.2 restrict to an adjoint pair between $\mathbb{K}_m^c(\mathrm{Proj}\,X)$ and $\mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)^{\mathrm{op}}$ by showing that $U(-)^{\circ}$ sends compact objects to complexes with bounded coherent cohomology, and vice versa for $(-)^{\circ}U_{\lambda}$. In part (B) we prove that the unit and counit of this adjunction are isomorphisms, so the two functors involved are actually equivalences.

(A) Given a compact object $\mathscr{P}$ in $\mathbb{K}_m(\operatorname{Proj} X)$, proving that $\mathscr{P}^{\circ}$ has bounded coherent cohomology is, by Proposition 6.12, a local question. Locally we are working in $\mathbb{K}_m(\operatorname{Proj} A)$ for a noetherian ring A, where a compact object is by [Nee06a, Proposition 6.12] a bounded below complex P of finitely generated projectives with $H^i \operatorname{Hom}_A(P, A) = 0$ for $i \ll 0$. So in this case it is clear, using noetherianness of A, that $\mathbb{R}\mathrm{Flat}(P, A) \cong \operatorname{Hom}_A(P, A)$ has bounded coherent cohomology (here we use Lemma 6.6).

Next, we show that for a bounded complex $\mathscr{G}$ of coherent sheaves with $\mathbb{K}$-flat resolution $\mathscr{P}$, the dual $\mathscr{P}^{\circ}$ is compact. By Lemma 3.15 and Proposition 6.12 this is a local question. Given a noetherian ring A let G be a bounded complex of finitely generated A-modules and P a projective resolution of G by finitely generated projectives; the dual $\mathbb{R}\mathrm{Flat}(P, A)$ is compact in $\mathbb{K}_m(\operatorname{Proj} A)$ by Lemma 6.6 and [Nee06a, Proposition 6.12]. This proves that the adjunction of Proposition 7.2 restricts to an adjunction

$$\mathbb{K}_m^c(\operatorname{Proj} X) \underset{(-)^{\circ}U_{\lambda}}{\overset{U(-)^{\circ}}{\rightleftarrows}} \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)^{\mathrm{op}} \tag{7.5}$$

(B) Given a complex $\mathscr{F}$ that is compact in $\mathbb{K}_m(\operatorname{Proj} X)$, the counit $e : U_{\lambda}U(\mathscr{F}^{\circ}) \longrightarrow \mathscr{F}^{\circ}$ of the adjunction between U and U_{λ} is a $\mathbb{K}$-flat resolution, by Remark 5.9. But by Lemma 7.3 the complex $\mathscr{F}^{\circ}$ is already $\mathbb{K}$-flat, so e is an isomorphism in $\mathbb{K}_m(\operatorname{Proj} X)$. The unit of the adjunction (7.5) is the composite

$$\mathscr{F} \longrightarrow \mathscr{F}^{\circ\circ} \xrightarrow{e^{\circ}} (U_{\lambda}U(\mathscr{F}^{\circ}))^{\circ}$$

of two isomorphisms, by Lemma 7.3 (recall that compact objects are locally complexes of vector bundles by Lemma 6.10). This proves that the unit of (7.5) is a natural equivalence.

Given a bounded complex $\mathscr{G}$ of coherent sheaves the unit morphism $f : \mathscr{G} \longrightarrow UU_{\lambda}(\mathscr{G})$ is an isomorphism in $\mathbb{D}(\mathfrak{Qco}X)$, as the left adjoint U_{λ} is fully faithful. Moreover, the $\mathbb{K}$-flat resolution $U_{\lambda}(\mathscr{G})$ is locally a complex of vector bundles (Lemma 6.11) so the counit of the adjunction (7.5) is the composite

$$\mathscr{G} \xrightarrow{f} UU_{\lambda}(\mathscr{G}) \longrightarrow U(U_{\lambda}(\mathscr{G})^{\circ\circ})$$

of two isomorphisms; see Lemma 7.3. This proves that the counit of (7.5) is a natural equivalence, and establishes that $U(-)^{\circ}$ is an equivalence with quasi-inverse $(-)^{\circ}U_{\lambda}$. □

We say that X *has enough vector bundles* if every coherent sheaf can be written as the quotient of a vector bundle; in this case, the theorem simplifies. Note that this condition is satisfied by any quasi-projective variety or, more generally, by any scheme with an ample family of line bundles; see [TT90, Lemma 2.1.3].

Remark 7.5. Assume that X has enough vector bundles, so that every bounded complex $\mathscr{G}$ of coherent sheaves has a resolution $\mathscr{V} \longrightarrow \mathscr{G}$ by a bounded above complex $\mathscr{V}$ of vector bundles. This is a $\mathbb{K}$-flat resolution, and by Corollary 6.13 we have

$$\mathscr{V}^{\circ} = \mathscr{H}om(\mathscr{V}, \mathcal{O}_X)$$

so the equivalence $(-)^{\circ}U_\lambda$ of Theorem 7.4 identifies $\mathscr{G}$ with the sheaf dual $\mathscr{H}om(\mathscr{V}, \mathcal{O}_X)$ of its resolution by vector bundles. The theorem also implies that every compact object in $\mathbb{K}_m(\operatorname{Proj} X)$ is of this form.

Up to isomorphism the coherent sheaves form a set that generates the bounded derived category of coherent sheaves. From the theorem we learn that the Spanier-Whitehead duals of flat resolutions of elements of this set generate $\mathbb{K}_m^c(\operatorname{Proj} X)$ as a triangulated category; such resolutions exist, and are unique in $\mathbb{K}_m(\operatorname{Proj} X)$, by Remark 5.9. This constructs an explicit compact generating set, as described in the following.

Corollary 7.6. *The category $\mathbb{K}_m(\operatorname{Proj} X)$ is compactly generated, and*

$$\{\Sigma^i \mathscr{P}_{\mathscr{G}}^{\circ} \,|\, \mathscr{G} \text{ is a coherent sheaf and } i \in \mathbb{Z}\}$$

is a compact generating set, where $\mathscr{P}_{\mathscr{G}}$ denotes a resolution by flat quasi-coherent sheaves

$$\cdots \longrightarrow \mathscr{P}_{\mathscr{G}}^{-2} \longrightarrow \mathscr{P}_{\mathscr{G}}^{-1} \longrightarrow \mathscr{P}_{\mathscr{G}}^{0} \longrightarrow \mathscr{G} \longrightarrow 0$$

Our attention now turns to the mock stable derived category $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$. We know from the defining recollement (Theorem 5.5) that this category is equivalent to the quotient of $\mathbb{K}_m(\operatorname{Proj} X)$ by the derived category $\mathbb{D}(\mathfrak{Qco} X)$, identified with the subcategory of $\mathbb{K}$-flat resolutions in $\mathbb{K}_m(\operatorname{Proj} X)$. A classification of the compact objects in $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ will therefore follow from the Neeman-Ravenel-Thomason localization theorem. As part of the proof, we will need the following comparison of function objects. Recall that the function object $\mathbb{R}\mathscr{H}om_{\mathrm{qc}}(-,-)$ in $\mathbb{D}(\mathfrak{Qco} X)$ was defined in Section 6.1.

Lemma 7.7. *There is a canonical morphism in $\mathbb{D}(\mathfrak{Qco} X)$ natural in both variables*

$$\theta : \mathbb{R}\mathscr{F}lat(\mathscr{P}, \mathscr{F}) \longrightarrow \mathbb{R}\mathscr{H}om_{\mathrm{qc}}(\mathscr{P}, \mathscr{F})$$

which is an isomorphism in $\mathbb{D}(\mathfrak{Qco} X)$ if $\mathscr{P}$ is $\mathbb{K}$-flat.

Proof. If we agree to write "$\operatorname{Hom}_{\mathbb{K}_m}(-,-)$" for $\operatorname{Hom}_{\mathbb{K}_m(\operatorname{Proj} X)}(-,-)$ and "$\operatorname{Hom}_{\mathbb{D}}(-,-)$" for $\operatorname{Hom}_{\mathbb{D}(\mathfrak{Qco} X)}(-,-)$ then we have a natural morphism

$$\begin{aligned}
\operatorname{Hom}_{\mathbb{D}}(\mathscr{A}, \mathbb{R}\mathscr{F}lat(\mathscr{P}, \mathscr{F})) &\xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}_m}(U_\lambda \mathscr{A}, \mathbb{R}\mathscr{F}lat(\mathscr{P}, \mathscr{F})) && \text{(Adjunction)}\\
&\xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}_m}(U_\lambda(\mathscr{A}) \otimes \mathscr{P}, \mathscr{F}) && \text{(Adjunction)}\\
&\longrightarrow \operatorname{Hom}_{\mathbb{D}}(U_\lambda(\mathscr{A}) \otimes \mathscr{P}, \mathscr{F}) && \text{(Quotient) } (\star)\\
&\xrightarrow{\sim} \operatorname{Hom}_{\mathbb{D}}(\mathscr{A} \underset{=}{\otimes} \mathscr{P}, \mathscr{F}) && \text{(Definition)}\\
&\xrightarrow{\sim} \operatorname{Hom}_{\mathbb{D}}(\mathscr{A}, \mathbb{R}\mathscr{H}om_{\mathrm{qc}}(\mathscr{P}, \mathscr{F})) && \text{(Adjunction)}
\end{aligned}$$

which yields a canonical morphism $\theta : \mathbb{R}\mathscr{F}lat(\mathscr{P}, \mathscr{F}) \longrightarrow \mathbb{R}\mathscr{H}om_{\mathrm{qc}}(\mathscr{P}, \mathscr{F})$ in $\mathbb{D}(\mathfrak{Qco} X)$. If $\mathscr{P}$ is $\mathbb{K}$-flat then the tensor product $U_\lambda(\mathscr{A}) \otimes \mathscr{P}$ is $\mathbb{K}$-flat and the step marked $(\star)$ is an isomorphism, by Remark 5.9(i), so we conclude that θ is an isomorphism in $\mathbb{D}(\mathfrak{Qco} X)$. ☐

The Spanier-Whitehead dual on $\mathbb{D}(\mathfrak{Qco}X)$ is the usual derived dual

$$(-)^\vee = \mathbb{R}\mathscr{H}om_{\mathrm{qc}}(-, \mathcal{O}_X)$$

Recall that a complex of quasi-coherent sheaves is *perfect* if it is locally isomorphic, in the derived category of quasi-coherent sheaves, to a bounded complex of vector bundles. These are the compact objects in $\mathbb{D}(\mathfrak{Qco}X)$ [Nee96, Proposition 2.5] and for the sake of having emotive notation we denote the full subcategory of perfect complexes by

$$\mathrm{Perf}(X) = \mathbb{D}^c(\mathfrak{Qco}X)$$

The perfect complexes are very well studied in the literature; the most complete accounts can be found in [SGA6, §I.4] or [TT90, §2]. In particular, a great deal is known about the interaction between perfect complexes and the function object $\mathbb{R}\mathscr{H}om(-,-)$ in $\mathbb{D}(X)$, and using Lemma 6.5 most of these results translate directly to $\mathbb{D}(\mathfrak{Qco}X)$. Properties of perfect complexes can also be viewed as consequences of the fact that $\mathbb{D}(\mathfrak{Qco}X)$ is a stable homotopy category; see [AJPV07] and [HPS97, Appendix A.2].

Lemma 7.8. *Let $\mathscr{C}$ be a perfect complex. There is an isomorphism in $\mathbb{K}_m(\mathrm{Proj}\, X)$*

$$U_\lambda(\mathscr{C}^\vee) \xrightarrow{\sim} (U_\lambda \mathscr{C})^\circ$$

Proof. These complexes are $\mathbb{K}$-flat by Lemma 7.3, and to show that they are isomorphic in $\mathbb{K}_m(\mathrm{Proj}\, X)$ it suffices, by Remark 5.9(i), to prove that they are isomorphic in $\mathbb{D}(\mathfrak{Qco}X)$. This is a consequence of Lemma 7.7. □

In the next theorem we identify compact objects in $\mathbb{K}_{m,\mathrm{ac}}(\mathrm{Proj}\, X)$ with objects of the triangulated category of singularities [Orl04]

$$\mathbb{D}^b_{\mathrm{sg}}(X) = \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X) / \mathrm{Perf}(X)$$

We have described compact objects in the mock homotopy category of projectives as the duals of resolutions, and something similar is true for $\mathbb{K}_{m,\mathrm{ac}}(\mathrm{Proj}\, X)$. However, to give the exact statement it would be necessary to develop the properties of complete flat resolutions over schemes, so we will give it elsewhere.

Theorem 7.9. *There is a canonical equivalence up to direct factors*

$$\mathbb{D}^b_{\mathrm{sg}}(X)^{op} \xrightarrow{\sim} \mathbb{K}^c_{m,\mathrm{ac}}(\mathrm{Proj}\, X)$$

Proof. By Theorem 5.5 we have a recollement

$$\mathbb{K}_{m,\mathrm{ac}}(\mathrm{Proj}\, X) \rightleftarrows \mathbb{K}_m(\mathrm{Proj}\, X) \rightleftarrows \mathbb{D}(\mathfrak{Qco}X)$$

in which $\mathbb{K}_m(\mathrm{Proj}\, X)$ is compactly generated (Theorem 4.10) and $\mathbb{D}(\mathfrak{Qco}X)$ is compactly generated [Nee96, Proposition 2.5]. The inclusion $I : \mathbb{K}_{m,\mathrm{ac}}(\mathrm{Proj}\, X) \longrightarrow \mathbb{K}_m(\mathrm{Proj}\, X)$ has left adjoint I_λ and applying the Neeman-Ravenel-Thomason localization theorem (in the form of Corollary 2.10) we deduce that the restricted functor

$$I_\lambda : \mathbb{K}^c_m(\mathrm{Proj}\, X) \longrightarrow \mathbb{K}^c_{m,\mathrm{ac}}(\mathrm{Proj}\, X)$$

induces an equivalence up to direct factors $\mathbb{K}^c_m(\operatorname{Proj} X)/\operatorname{Perf}(X) \xrightarrow{\sim} \mathbb{K}^c_{m,\mathrm{ac}}(\operatorname{Proj} X)$. It remains to identify this quotient with the one occurring in the statement of the theorem. By Theorem 7.4 we have an equivalence

$$(-)^\circ U_\lambda : \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco} X)^{\mathrm{op}} \xrightarrow{\sim} \mathbb{K}^c_m(\operatorname{Proj} X)$$

which we claim identifies the subcategory $\operatorname{Perf}(X)^{\mathrm{op}}$ of the left hand side with the copy of $\operatorname{Perf}(X)$ existing in $\mathbb{K}^c_m(\operatorname{Proj} X)$ as the essential image of U_λ on compact objects. To prove this, let $\mathscr{C}$ be a perfect complex. Then by Lemma 7.8 the Spanier-Whitehead dual $(U_\lambda \mathscr{C})^\circ$ is the $\mathbb{K}$-flat resolution of the perfect complex $\mathscr{C}^\vee$, thus an object of the image of U_λ on compacts. Conversely, we have

$$U_\lambda(\mathscr{C}) \cong U_\lambda(\mathscr{C}^{\vee\vee}) \cong U(\mathscr{C}^\vee)^\circ$$

so every object in the image of U_λ on compacts is of the form $U_\lambda(\mathscr{D})^\circ$ for some perfect complex $\mathscr{D}$, proving the claim. It follows that there is an equivalence

$$\begin{aligned}(\mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco} X)/\operatorname{Perf}(X))^{\mathrm{op}} &= \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco} X)^{\mathrm{op}}/\operatorname{Perf}(X)^{\mathrm{op}} \\ &\cong \mathbb{K}^c_m(\operatorname{Proj} X)/\operatorname{Perf}(X)\end{aligned}$$

which completes the proof of the theorem. $\square$

Chapter 8

The Infinite Completion of Grothendieck Duality

Let X be a noetherian scheme with a dualizing complex. We argued in the introduction to this thesis that the mock homotopy category $\mathbb{K}_m(\operatorname{Proj} X)$ of projectives and the homotopy category $\mathbb{K}(\operatorname{Inj} X)$ of injectives should be understood as extensions of the derived category of quasi-coherent sheaves $\mathbb{D}(\mathfrak{Qco} X)$, which adjoin acyclic complexes of interest. We prove in this chapter that the dualizing complex $\mathscr{D}$, assumed to be a bounded complex of injective quasi-coherent sheaves, induces an equivalence of these two extensions (Theorem 8.4)

$$- \otimes \mathscr{D} : \mathbb{K}_m(\operatorname{Proj} X) \xrightarrow{\sim} \mathbb{K}(\operatorname{Inj} X) \tag{8.1}$$

Restricting to compact objects recovers the equivalence of Grothendieck duality

$$\mathbb{R}\mathscr{H}om_{\mathrm{qc}}(-, \mathscr{D}) : \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco} X)^{\mathrm{op}} \xrightarrow{\sim} \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco} X) \tag{8.2}$$

where $\mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco} X)$ is the bounded derived category of coherent sheaves (see Remark 7.1). For affine schemes this result is due to Iyengar and Krause [IK06], and we refer the reader to the introduction of this thesis for a discussion of their work. We will not try to survey the literature on Grothendieck duality, as the subject is too vast, but the reader can find a very good introduction in Conrad's [Con00]. For us, the central object is the *dualizing complex* [Har66, §V.2]. This is a bounded below complex $\mathscr{D}$ in $\mathbb{D}(X)$ with coherent cohomology and finite injective dimension, such that the canonical morphism

$$\mathscr{F} \longrightarrow \mathbb{R}\mathscr{H}om(\mathbb{R}\mathscr{H}om(\mathscr{F}, \mathscr{D}), \mathscr{D}) \tag{8.3}$$

is an isomorphism in $\mathbb{D}(X)$ for every complex $\mathscr{F} \in \mathbb{D}^b_{\mathrm{coh}}(X)$, which is to say, every complex of sheaves of modules with bounded coherent cohomology. We deduce from (8.3) that there is an equivalence (8.2), modulo one technical point; see Lemma 8.1 below.

Dualizing complexes are very useful, and exist for a large class of schemes. For example, any scheme of finite type over a field (and thus any variety) admits a dualizing complex [Har66, §II.10]. A dualizing complex is quasi-isomorphic to a bounded complex of injective quasi-coherent sheaves [Har66, II 7.20$(i)_{qc}$]. Hence, if X admits a dualizing complex, it admits a dualizing complex that is a bounded complex of injective quasi-coherent sheaves, so we are free to adopt the following convention.

Setup. In this section X is a noetherian scheme with dualizing complex $\mathscr{D}$ and sheaves are defined over X by default. We will always assume that $\mathscr{D}$ is a bounded complex of injective quasi-coherent sheaves.

One technical point is that we want to work with $\mathbb{D}^b_{\text{coh}}(\mathfrak{Qco}X)$ rather than $\mathbb{D}^b_{\text{coh}}(X)$, in order to deal throughout with quasi-coherent sheaves; see Remark 7.1 for an explanation of the notation and Chapter 6 for a discussion of the function objects in $\mathbb{D}(\mathfrak{Qco}X)$, denoted here by $\mathbb{R}\mathscr{H}om_{\text{qc}}(-,-)$. The next lemma consists of replacing one version of the bounded derived category by another in the defining property of a dualizing complex.

Lemma 8.1. *There is an equivalence of triangulated categories*

$$\mathbb{R}\mathscr{H}om_{\text{qc}}(-,\mathscr{D}) : \mathbb{D}^b_{\text{coh}}(\mathfrak{Qco}X)^{op} \xrightarrow{\sim} \mathbb{D}^b_{\text{coh}}(\mathfrak{Qco}X)$$

Proof. This is a consequence of Lemma 6.5 and the definition of the dualizing complex. □

We need to explain why tensoring with $\mathscr{D}$ sends complexes of flat sheaves to complexes of injective sheaves. This functor should also send acyclic $\mathbb{K}$-flat complexes to contractible complexes in order to be well-defined on $\mathbb{K}_m(\operatorname{Proj} X)$. Note that the next lemma applies just as well to an arbitrary complex $\mathscr{D}$ of injective quasi-coherent sheaves.

Lemma 8.2. *If $\mathscr{F}$ is a complex of flat quasi-coherent sheaves then $\mathscr{F}\otimes\mathscr{D}$ is in $\mathbb{K}(\operatorname{Inj} X)$. If moreover $\mathscr{F}$ belongs to $\mathbb{E}(X)$ then $\mathscr{F}\otimes\mathscr{D}$ is contractible.*

Proof. We have $(\mathscr{F}\otimes\mathscr{D})^n = \oplus_{i+j=n}\mathscr{F}^i\otimes\mathscr{D}^j$, and an arbitrary coproduct of injectives in $\mathfrak{Qco}(X)$ is injective, so we reduce to the case where $\mathscr{F}$ and $\mathscr{D}$ are single quasi-coherent sheaves. The claim can now be checked on stalks [Har66, Proposition II.7.17] so we reduce to showing that $F\otimes_A D$ is injective for a noetherian ring A, flat A-module F and injective A-module D. By Lazard's thesis [Laz64], F is the direct limit of a family of finitely generated free modules. The functor $-\otimes_A D$ commutes with colimits and any direct limit of injective A-modules is injective [Mat58] so we reduce to the case of F free of finite rank, which is trivial.

Now suppose that $\mathscr{F}$ belongs to $\mathbb{E}(X)$, so that it is acyclic and $\mathbb{K}$-flat. We prove that the complex $\mathscr{F}\otimes\mathscr{D}$ is contractible. It is enough to show that it is acyclic and has injective kernels, both of which can be checked on stalks, so we can take $X = \mathit{Spec}(A)$ affine. For a noetherian ring A, acyclic $\mathbb{K}$-flat complex F of flat A-modules and complex D of injective A-modules we have to prove that $D\otimes_A F$ is contractible. The proof of this statement is given in [Nee06a, Corollary 8.7]. □

Tensoring with $\mathscr{D}$ defines a triangulated functor from $\mathbb{K}(\mathfrak{Qco}X)$ to itself. By the lemma this functor sends $\mathbb{K}(\operatorname{Flat} X)$ into $\mathbb{K}(\operatorname{Inj} X)$, and the restriction $\mathbb{K}(\operatorname{Flat} X) \longrightarrow \mathbb{K}(\operatorname{Inj} X)$ vanishes on $\mathbb{E}(X)$. We deduce a triangulated functor out of the Verdier quotient

$$-\otimes\mathscr{D} : \mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{K}(\operatorname{Inj} X) \tag{8.4}$$

We will prove that this is an equivalence, but first we need to understand how to relate the tensor product $-\otimes \mathscr{D}$ and function object $\mathbb{R}\mathscr{H}om(-,\mathscr{D})$.

Lemma 8.3. *Let $\mathscr{F}$ be a complex of flat quasi-coherent sheaves that is locally a complex of vector bundles. There is a natural isomorphism in $\mathbb{K}(X)$*

$$\pi : \mathscr{F}^\circ \otimes \mathscr{D} \xrightarrow{\sim} \mathscr{H}om(\mathscr{F},\mathscr{D})$$

If further $\mathscr{F}$ is $\mathbb{K}$-flat then $\mathscr{F}^\circ \otimes \mathscr{D}$ is $\mathbb{K}$-injective.

Proof. To be clear on some notation: in Chapter 6 we defined the closed monoidal structure on $\mathbb{K}_m(\operatorname{Proj} X)$ and in particular the Spanier-Whitehead dual $(-)^\circ$. We also explained in Definition 6.9 what it means for a complex of flat quasi-coherent sheaves to be locally a complex of vector bundles. Applying $-\otimes\mathscr{D}$ to the canonical morphism $\mathscr{F}^\circ \otimes \mathscr{F} \longrightarrow \mathcal{O}_X$ in $\mathbb{K}_m(\operatorname{Proj} X)$ we obtain a morphism

$$(\mathscr{F}^\circ \otimes \mathscr{D}) \otimes \mathscr{F} \cong (\mathscr{F}^\circ \otimes \mathscr{F}) \otimes \mathscr{D} \longrightarrow \mathscr{D}$$

which is adjoint to the desired morphism π in $\mathbb{K}(X)$. Suppose we can show that π restricts to an isomorphism in $\mathbb{K}(U)$ for every affine open subset $U \subseteq X$. Then the mapping cone $\mathscr{I} = \operatorname{cone}(\pi)$ is a complex of injective (not necessarily quasi-coherent) sheaves contractible on every open affine. For any affine open cover $\mathfrak{U} = \{U_0, \ldots, U_d\}$ the Čech resolution

$$0 \longrightarrow \mathscr{I} \longrightarrow \mathscr{C}^0(\mathfrak{U},\mathscr{I}) \longrightarrow \mathscr{C}^1(\mathfrak{U},\mathscr{I}) \longrightarrow \cdots \longrightarrow \mathscr{C}^d(\mathfrak{U},\mathscr{I}) \longrightarrow 0$$

decomposes into short exact sequences of complexes of injective sheaves, each degree-wise split. From the corresponding triangles in $\mathbb{K}(X)$ and the local vanishing of $\mathscr{I}$ we deduce that $\mathscr{I}$ vanishes in $\mathbb{K}(X)$, which implies that π is an isomorphism in $\mathbb{K}(X)$.

Because $\mathscr{F}$ is locally a complex of vector bundles, we can apply Proposition 6.12 to reduce to the case where $X = \mathit{Spec}(A)$ for a noetherian ring A with dualizing complex D (as always, assumed to be a bounded complex of injectives) and $\mathscr{F}$ is replaced by a complex F of finitely generated projectives. In this case we have to show that the morphism

$$\pi : \mathbb{R}\mathrm{Flat}(F, A) \otimes_A D \longrightarrow \operatorname{Hom}_A(F, D)$$

is an isomorphism in $\mathbb{K}(A)$. Happily, we are in the situation where $\mathbb{R}\mathrm{Flat}(-,-)$ simplifies. By Lemma 6.6 there is an isomorphism $\operatorname{Hom}_A(F,A) \cong \mathbb{R}\mathrm{Flat}(F,A)$ in $\mathbb{K}_m(\operatorname{Proj} A)$ and we have reduced to checking that the canonical morphism

$$\operatorname{Hom}_A(F, A) \otimes_A D \longrightarrow \operatorname{Hom}_A(F, D)$$

is an isomorphism in $\mathbb{K}(A)$, which is straightforward. It remains to prove the second claim. If $\mathscr{F}$ is $\mathbb{K}$-flat then $\mathscr{H}om(\mathscr{F},\mathscr{D})$ is $\mathbb{K}$-injective, and thus the homotopy equivalent complex $\mathscr{F}^\circ \otimes \mathscr{D}$ must also be $\mathbb{K}$-injective. $\square$

Combining our major results, we have the infinite completion of Grothendieck duality.

Theorem 8.4. *There is an equivalence of triangulated categories*

$$- \otimes \mathscr{D} : \mathbb{K}_m(\operatorname{Proj} X) \xrightarrow{\sim} \mathbb{K}(\operatorname{Inj} X)$$

making the following diagram commute up to natural equivalence

$$\begin{tikzcd}
\mathbb{K}_m^c(\operatorname{Proj} X) \arrow[rr, "-\otimes\mathscr{D}"] & & \mathbb{K}^c(\operatorname{Inj} X) \arrow[d, "\wr"', "\mathrm{can}"] \\
\mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)^{op} \arrow[u, "(-)^\circ U_\lambda", "\wr"'] \arrow[rr, "\sim", "\mathbb{R}\mathscr{H}om_{\mathrm{qc}}(-,\mathscr{D})"'] & & \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)
\end{tikzcd} \tag{8.5}$$

Proof. We begin by reminding the reader of the functors in the diagram (8.5). By Theorem 5.5 the canonical functor $U : \mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{D}(\mathfrak{Qco}X)$ has a left adjoint U_λ that takes $\mathbb{K}$-flat resolutions. The left side of (8.5) comes from Theorem 7.4, which proves that there is an equivalence

$$(-)^\circ U_\lambda : \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)^{\mathrm{op}} \xrightarrow{\sim} \mathbb{K}_m^c(\operatorname{Proj} X)$$

The right side of (8.5) comes from [Kra05, Proposition 2.3], where Krause shows that the canonical functor $\mathbb{K}(\operatorname{Inj} X) \longrightarrow \mathbb{D}(\mathfrak{Qco}X)$ induces an equivalence

$$\mathbb{K}^c(\operatorname{Inj} X) \xrightarrow{\sim} \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)$$

The bottom side of (8.5) is the defining property of the dualizing complex (Lemma 8.1). For the top to be well-defined, we have to prove that the functor $- \otimes \mathscr{D}$ of (8.4) preserves compactness. Consider the diagram

$$\begin{tikzcd}
\mathbb{K}_m(\operatorname{Proj} X) \arrow[rr, "-\otimes\mathscr{D}"] & & \mathbb{K}(\operatorname{Inj} X) \arrow[d, "\mathrm{can}"] \\
\mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)^{\mathrm{op}} \arrow[u, "(-)^\circ U_\lambda"] \arrow[rr, "\mathbb{R}\mathscr{H}om_{\mathrm{qc}}(-,\mathscr{D})"'] & & \mathbb{D}(\mathfrak{Qco}X)
\end{tikzcd}$$

The two ways around this diagram are naturally equivalent, as follows

$$\begin{aligned}
\{U_\lambda(-)\}^\circ \otimes \mathscr{D} &\xrightarrow{\sim} \mathscr{H}om(U_\lambda(-), \mathscr{D}) && \text{(Lemma 6.11 and Lemma 8.3)} \\
&\xrightarrow{\sim} \mathbb{R}\mathscr{H}om(U_\lambda(-), \mathscr{D}) && (\mathscr{D} \text{ is } \mathbb{K}\text{-injective}) \\
&\xrightarrow{\sim} \mathbb{R}\mathscr{H}om_{\mathrm{qc}}(-, \mathscr{D}) && \text{(Lemma 6.5)}
\end{aligned}$$

Let a compact object in $\mathbb{K}_m(\operatorname{Proj} X)$ be given. By Theorem 7.4 we can assume that this compact object is of the form $U_\lambda(\mathscr{G})^\circ$ for some $\mathscr{G}$ in $\mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)$. The resolution $U_\lambda(\mathscr{G})$ is locally a complex of vector bundles and is $\mathbb{K}$-flat, so Lemma 8.3 implies that $U_\lambda(\mathscr{G})^\circ \otimes \mathscr{D}$ is $\mathbb{K}$-injective. From the following isomorphism in $\mathbb{D}(\mathfrak{Qco}X)$

$$U_\lambda(\mathscr{G})^\circ \otimes \mathscr{D} \xrightarrow{\sim} \mathbb{R}\mathscr{H}om_{\mathrm{qc}}(\mathscr{G}, \mathscr{D})$$

we learn that $U_\lambda(\mathscr{G})^\circ \otimes \mathscr{D}$ has bounded coherent cohomology, and is consequently compact in $\mathbb{K}(\operatorname{Inj} X)$ by Krause's classification [Kra05, Proposition 2.3]. This proves that $- \otimes \mathscr{D}$ preserves compactness, and that the induced functor

$$- \otimes \mathscr{D} : \mathbb{K}_m^c(\operatorname{Proj} X) \longrightarrow \mathbb{K}^c(\operatorname{Inj} X) \tag{8.6}$$

fits into a square (8.5) commuting up to natural equivalence. Since every other side of this square is an equivalence, we infer that (8.6) is an equivalence, and a standard argument (Proposition 2.11) now allows us to conclude that $- \otimes \mathscr{D} : \mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{K}(\operatorname{Inj} X)$ is an equivalence, because it is a coproduct preserving triangulated functor between compactly generated triangulated categories (see Theorem 4.10 and [Kra05, Proposition 2.3]) which restricts to an equivalence on compact objects. □

Let us explain the significance of the commutative diagram (8.5) of the theorem.

Remark 8.5. Bounded complexes of coherent sheaves can be viewed as compact objects in both $\mathbb{K}_m(\operatorname{Proj} X)$ and $\mathbb{K}(\operatorname{Inj} X)$, in the former category by taking the Spanier-Whitehead dual of a $\mathbb{K}$-flat resolution, and in the latter by taking a $\mathbb{K}$-injective resolution; see Chapter 7. Given a bounded complex $\mathscr{G}$ of coherent sheaves, the equivalence

$$- \otimes \mathscr{D} : \mathbb{K}_m(\operatorname{Proj} X) \xrightarrow{\sim} \mathbb{K}(\operatorname{Inj} X) \tag{8.7}$$

does *not* interchange the two compact objects corresponding to $\mathscr{G}$. It sends the compact object of $\mathbb{K}_m(\operatorname{Proj} X)$ determined by $\mathscr{G}$ to the compact object of $\mathbb{K}(\operatorname{Inj} X)$ determined by its Grothendieck dual $\mathbb{R}\mathscr{H}om_{\mathrm{qc}}(\mathscr{G}, \mathscr{D})$. To be precise, writing $\mathbf{p}(-)$ for a $\mathbb{K}$-flat resolution and $\mathbf{i}(-)$ for a $\mathbb{K}$-injective resolution, we have an isomorphism in $\mathbb{K}(\operatorname{Inj} X)$

$$(\mathbf{p}\mathscr{G})^\circ \otimes \mathscr{D} \xrightarrow{\sim} \mathbf{i}\,\mathbb{R}\mathscr{H}om_{\mathrm{qc}}(\mathscr{G}, \mathscr{D})$$

This is the content of (8.5) and the sense in which (8.7) extends Grothendieck duality.

Remark 8.6. We have recollements, by Theorem 5.5 and [Kra05, Corollary 4.2]

$$\begin{array}{c} \mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \rightleftarrows \mathbb{K}_m(\operatorname{Proj} X) \rightleftarrows \mathbb{D}(\mathfrak{Qco} X) \\ \mathbb{K}_{\mathrm{ac}}(\operatorname{Inj} X) \rightleftarrows \mathbb{K}(\operatorname{Inj} X) \rightleftarrows \mathbb{D}(\mathfrak{Qco} X) \end{array}$$

One can ask whether the equivalence of Theorem 8.4 identifies the recollements, in the sense that it sends acyclic complexes to acyclic complexes. In fact, this can only happen when X is a Gorenstein scheme: if the functor $- \otimes \mathscr{D}$ identifies the subcategories of acyclic complexes, then it identifies their orthogonals, and we have an equivalence

$$- \underline{\underline{\otimes}}\, \mathscr{D} : \mathbb{D}(\mathfrak{Qco} X) \xrightarrow{\sim} {}^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \xrightarrow{\sim} {}^{\perp}\mathbb{K}_{\mathrm{ac}}(\operatorname{Inj} X) \xrightarrow{\sim} \mathbb{D}(\mathfrak{Qco} X)$$

This equivalence must preserve compactness, so $\mathscr{D} \cong \mathscr{D} \underline{\underline{\otimes}}\, \mathcal{O}_X$ is perfect and we can conclude that X is Gorenstein [Har66, V.9.1].

Next we study the quasi-inverse of the equivalence in Theorem 8.4, using the function object $\mathscr{H}om_{\mathrm{qc}}(-,-)$ in the homotopy category $\mathbb{K}(\mathfrak{Qco} X)$, as defined in Section 6.1.

Lemma 8.7. *If $\mathscr{I}, \mathscr{I}'$ are injective quasi-coherent sheaves then $\mathscr{H}om_{\mathrm{qc}}(\mathscr{I}, \mathscr{I}')$ is flat.*

Proof. Let $\mathfrak{U} = \{U_0, \dots, U_d\}$ be an affine open cover of X and consider the Čech resolution

$$0 \longrightarrow \mathscr{I}' \longrightarrow \mathscr{C}^0(\mathfrak{U}, \mathscr{I}') \longrightarrow \cdots \longrightarrow \mathscr{C}^d(\mathfrak{U}, \mathscr{I}') \longrightarrow 0$$

This is an exact sequence of injective quasi-coherent sheaves that decomposes into a series of short, *split* exact sequences. The functor $\mathscr{H}om_{\mathrm{qc}}(\mathscr{I}, -)$ preserves split exact sequences, so to prove that $\mathscr{H}om_{\mathrm{qc}}(\mathscr{I}, \mathscr{I}')$ is flat it suffices to show that $\mathscr{H}om_{\mathrm{qc}}(\mathscr{I}, f_*(\mathscr{I}'|_U))$ is flat whenever $f : U \longrightarrow X$ is the inclusion of an affine open subset. In this case there is an isomorphism of sheaves (using adjointness, see Lemma 6.8)

$$\mathscr{H}om_{\mathrm{qc}}(\mathscr{I}, f_*(\mathscr{I}'|_U)) \cong f_*\mathscr{H}om_{\mathrm{qc}}(\mathscr{I}|_U, \mathscr{I}'|_U)$$

The functor f_* preserves flatness of quasi-coherent sheaves, so we have reduced to the case where $X = Spec(A)$ is affine, and $\mathscr{H}om_{\mathrm{qc}}(\mathscr{I}, \mathscr{I}') \cong \mathrm{Hom}_A(I, I')^{\sim}$ for some injective modules I, I'. This module is flat, by a standard argument. $\square$

Given a complex $\mathscr{I}$ of injective quasi-coherent sheaves the complex $\mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, \mathscr{I})$ is, in degree $n \in \mathbb{Z}$, the following product in the category $\mathfrak{Qco}(X)$ of quasi-coherent sheaves

$$\mathscr{H}om^n_{\mathrm{qc}}(\mathscr{D}, \mathscr{I}) = \prod_{q \in \mathbb{Z}}^{\mathrm{qc}} \mathscr{H}om_{\mathrm{qc}}(\mathscr{D}^q, \mathscr{I}^{q+n})$$

Because $\mathscr{D}$ is bounded this is a finite direct sum of flat sheaves, which is flat, so we have defined a triangulated functor $\mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, -) : \mathbb{K}(\mathrm{Inj}\, X) \longrightarrow \mathbb{K}(\mathrm{Flat}\, X)$. Composing with the quotient $\mathbb{K}(\mathrm{Flat}\, X) \longrightarrow \mathbb{K}_m(\mathrm{Proj}\, X)$ defines a triangulated functor

$$\mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, -) : \mathbb{K}(\mathrm{Inj}\, X) \longrightarrow \mathbb{K}_m(\mathrm{Proj}\, X)$$

We show in the next proposition that this is an equivalence.

Lemma 8.8. *Let $\mathscr{I}$ be a complex of injective quasi-coherent sheaves. Then $\mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, \mathscr{I})$ belongs to the orthogonal $\mathbb{E}(X)^{\perp}$ as an object of $\mathbb{K}(\mathrm{Flat}\, X)$.*

Proof. This follows from the adjunction between $- \otimes \mathscr{D}$ and $\mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, -)$ and the fact that $\mathscr{E} \otimes \mathscr{D}$ is contractible whenever $\mathscr{E}$ belongs to $\mathbb{E}(X)$ (Lemma 8.2). $\square$

Proposition 8.9. *There is a pair of equivalences of triangulated categories*

$$\begin{tikzcd} \mathbb{K}_m(\mathrm{Proj}\, X) \arrow[r, shift left, "- \otimes \mathscr{D}"] \arrow[r, phantom, "\sim"] & \mathbb{K}(\mathrm{Inj}\, X) \arrow[l, shift left, "\mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, -)"] \end{tikzcd}$$

each quasi-inverse to the other.

Proof. For a complex $\mathscr{F}$ of flat quasi-coherent sheaves and a complex $\mathscr{I}$ of injective quasi-coherent sheaves, there is a natural isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathbb{K}(\mathrm{Inj}\, X)}(\mathscr{F} \otimes \mathscr{D}, \mathscr{I}) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{K}(\mathrm{Flat}\, X)}(\mathscr{F}, \mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, \mathscr{I})) && \text{(Proposition 6.17)} \\ &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{K}_m(\mathrm{Proj}\, X)}(\mathscr{F}, \mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, \mathscr{I})) && \text{(Lemma 8.8)} \end{aligned}$$

This defines $\mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, -)$ and $- \otimes \mathscr{D}$ as an adjoint pair of functors between $\mathbb{K}_m(\operatorname{Proj} X)$ and $\mathbb{K}(\operatorname{Inj} X)$. We know from Theorem 8.4 that $- \otimes \mathscr{D}$ is an equivalence, which implies that $\mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, -)$ is an equivalence, and it must be the desired quasi-inverse. □

Using this equivalence we can define a closed monoidal structure on $\mathbb{K}(\operatorname{Inj} X)$. Let us tell the reader what the structure is over a noetherian ring A, but delay the proof until Appendix B where we can treat schemes on the same footing. Let D be the dualizing complex (which is, as always, a bounded complex of injectives) and observe that D is the unit object for the tensor product in the following closed monoidal structure.

Proposition 8.10. *The category* $\mathbb{K}(\operatorname{Inj} A)$ *is closed symmetric monoidal: it has a tensor product* $- \otimes_{\mathrm{Inj}} -$ *and function object* $\operatorname{Inj}(-,-)$ *defined by*

$$\begin{aligned} I \otimes_{\mathrm{Inj}} J &= I \otimes_A \operatorname{Hom}_A(D, J) \\ \operatorname{Inj}(I, J) &= D \otimes_A \operatorname{Hom}_A(I, J) \end{aligned}$$

which are compatible with the triangulation.

Proof. See Proposition B.6 and Remark B.7. □

Chapter 9

Applications

9.1 Local Cohomology

The local cohomology theory of Grothendieck [Har67] arises from a Bousfield localization of the derived category of quasi-coherent sheaves; see [BN93, §6]. In this section we give the analogue for the mock homotopy category of projectives.

To see the connection between local cohomology and Bousfield localization, let X be a scheme with quasi-compact open subset $U \subseteq X$ and inclusion $f : U \longrightarrow X$, and set $Z = X \setminus U$. For a quasi-coherent sheaf $\mathscr{F}$ there is an exact sequence

$$0 \longrightarrow \Gamma_Z(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow f_*(\mathscr{F}|_U) \tag{9.1}$$

where $\Gamma_Z(\mathscr{F})$ is the *subsheaf of sections with support in* Z. Passing to the derived category, we have for every complex $\mathscr{F}$ of quasi-coherent sheaves a triangle in $\mathbb{D}(\mathfrak{Qco}X)$

$$\mathbb{R}\Gamma_Z(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow \mathbb{R}f_*(\mathscr{F}|_U) \longrightarrow \Sigma\mathbb{R}\Gamma_Z(\mathscr{F}) \tag{9.2}$$

The next proposition realizes $\mathbb{R}\Gamma_Z(-)$ as one of the six functors in a recollement. To be precise, it is the right adjoint of the inclusion in $\mathbb{D}(\mathfrak{Qco}X)$ of the triangulated subcategory $\mathbb{D}_Z(\mathfrak{Qco}X) \subseteq \mathbb{D}(\mathfrak{Qco}X)$ of complexes with cohomology supported on Z.

Proposition 9.1. *There is a recollement*

$$\begin{tikzcd} \mathbb{D}(\mathfrak{Qco}U) \arrow[r, shift left=2] & \mathbb{D}(\mathfrak{Qco}X) \arrow[l, shift right=4] \arrow[l, shift left=4] \arrow[r, shift left=2] & \mathbb{D}_Z(\mathfrak{Qco}X) \arrow[l, shift right=4] \arrow[l, shift left=4] \end{tikzcd}$$

Proof. The restriction functor $\mathbb{D}(\mathfrak{Qco}X) \longrightarrow \mathbb{D}(\mathfrak{Qco}U)$ admits a fully faithful right adjoint $\mathbb{R}f_*$, so we have a colocalization sequence (Lemma 2.6), where B denotes the inclusion

$$\begin{tikzcd} \mathbb{D}(\mathfrak{Qco}U) \arrow[r, shift right, "\mathbb{R}f_*"'] & \mathbb{D}(\mathfrak{Qco}X) \arrow[l, shift right, "(-)|_U"'] \arrow[r, shift right, "B_\rho"'] & \mathbb{D}_Z(\mathfrak{Qco}X) \arrow[l, shift right, "B"'] \end{tikzcd}$$

The functor $\mathbb{R}f_*$ has a right adjoint; this is the Grothendieck duality theorem of Neeman [Nee96]. Using Lemma 2.3 we conclude that the pair $(B_\rho, \mathbb{R}f_*)$ is recollement. $\square$

Remark 9.2. Let $B : \mathbb{D}_Z(\mathfrak{QcoX}) \longrightarrow \mathbb{D}(\mathfrak{QcoX})$ be the inclusion, with right adjoint B_ρ. By the proposition, for any complex $\mathscr{F}$ of quasi-coherent sheaves there is a triangle

$$BB_\rho(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow \mathbb{R}f_*(\mathscr{F}|_U) \longrightarrow \Sigma BB_\rho(\mathscr{F}) \tag{9.3}$$

Let us explain why $B_\rho(-)$ is the local cohomology functor $\mathbb{R}\Gamma_Z(-)$ of Grothendieck. For simplicity, we assume that X is noetherian. If $\mathscr{F}$ is flasque then (9.1) is short exact, and thus defines a triangle in the derived category. Replacing $\mathscr{F}$ by its $\mathbb{K}$-injective resolution by injective quasi-coherent sheaves (which are flasque) we obtain a triangle (9.2). Comparing with (9.3) we deduce an isomorphism $B_\rho(\mathscr{F}) \cong \mathbb{R}\Gamma_Z(\mathscr{F})$, as claimed.

Setup. In this section X is a noetherian scheme and $U \subseteq X$ denotes an open subset with complement Z. We write $f : U \longrightarrow X$ for the inclusion.

Let $\mathbb{K}_{m,Z}(\operatorname{Proj} X)$ be the triangulated subcategory of $\mathbb{K}_m(\operatorname{Proj} X)$ consisting of those complexes that are "mock supported" on Z, in the sense that they are acyclic and $\mathbb{K}$-flat on the complement; see Definition 4.6. Next we give the analogue of local cohomology for the mock homotopy category.

Theorem 9.3. *There is a recollement*

$$\mathbb{K}_m(\operatorname{Proj} U) \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathbb{K}_m(\operatorname{Proj} X) \underset{\longleftarrow}{\overset{\longleftarrow}{\longrightarrow}} \mathbb{K}_{m,Z}(\operatorname{Proj} X)$$

Proof. We aim to copy the proof of Proposition 9.1, and in fact the argument is identical. However, in the present situation some work is required to show that the right adjoint of restriction, the analogue of $\mathbb{R}f_*$ in the earlier proof, is fully faithful.

The restriction functor $(-)|_U : \mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{K}_m(\operatorname{Proj} U)$ preserves coproducts and admits a right adjoint $\hat{\mathbb{R}}f_*$ by Brown representability. To prove that this functor is fully faithful it suffices, by a basic result of category theory, to prove that the counit

$$\varepsilon : (-)|_U \circ \hat{\mathbb{R}}f_* \longrightarrow 1$$

is a natural equivalence, and this is what we do. The idea is that $\hat{\mathbb{R}}f_*$ must be fully faithful "on" an open affine subset $W \subseteq U$ because, denoting by $g : W \longrightarrow U$ and $h : W \longrightarrow X$ the inclusions, we have a natural equivalence

$$\hat{\mathbb{R}}f_* \circ g_* \xrightarrow{\sim} h_*$$

and g_* and h_* are both fully faithful, by Lemma 4.7. Given a complex $\mathscr{F}$ of quasi-coherent sheaves on W, the counit $\varepsilon_{g_*(\mathscr{F})}$ is the isomorphism

$$\hat{\mathbb{R}}f_*(g_*(\mathscr{F}))|_U \cong h_*(\mathscr{F})|_U = g_*(\mathscr{F})$$

Hence the triangulated subcategory $\mathcal{L}$ of $\mathbb{K}_m(\operatorname{Proj} U)$ on which ε is an isomorphism contains the complexes defined over affine open subsets, and using Corollary 3.14 we conclude that ε is a natural equivalence. Thus $\hat{\mathbb{R}}f_*$ is fully faithful and, by Lemma 2.6, we have a colocalization sequence

$$\mathbb{K}_m(\operatorname{Proj} U) \underset{\hat{\mathbb{R}}f_*}{\overset{(-)|_U}{\rightleftarrows}} \mathbb{K}_m(\operatorname{Proj} X) \underset{}{\overset{\mathrm{inc}}{\rightleftarrows}} \mathbb{K}_{m,Z}(\operatorname{Proj} X)$$

Now we apply the machinery of compactly generated triangulated categories. Restriction preserves compactness, by Lemma 3.15, so the right adjoint $\hat{\mathbb{R}}f_*$ must preserve coproducts; see Lemma 2.9. But $\mathbb{K}_m(\operatorname{Proj} U)$ is compactly generated, so by Brown representability $\hat{\mathbb{R}}f_*$ has a right adjoint, and from Lemma 2.3 we deduce the desired recollement. $\square$

Bounded complexes of coherent sheaves correspond to compact objects of $\mathbb{K}_m(\operatorname{Proj} X)$, by Theorem 7.4. As one would expect, those complexes with cohomology supported on Z determine compact objects of $\mathbb{K}_{m,Z}(\operatorname{Proj} X)$. We introduce the following notation

$$\mathbb{D}^b_{\mathrm{coh},Z}(\mathfrak{Qco} X) := \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco} X) \cap \mathbb{D}_Z(\mathfrak{Qco} X)$$

for the triangulated subcategory of $\mathbb{D}(\mathfrak{Qco} X)$ whose objects are complexes with bounded coherent cohomology supported on Z. Our next result gives a classification of the compact objects in $\mathbb{K}_{m,Z}(\operatorname{Proj} X)$. We use the notation of Chapter 7, so that U_λ is the adjoint which calculates $\mathbb{K}$-flat resolutions and $(-)^\circ$ is the Spanier-Whitehead dual.

Corollary 9.4. *The triangulated category $\mathbb{K}_{m,Z}(\operatorname{Proj} X)$ is compactly generated and there is an equivalence*

$$U(-)^\circ : \mathbb{K}^c_{m,Z}(\operatorname{Proj} X) \xrightarrow{\sim} \mathbb{D}^b_{\mathrm{coh},Z}(\mathfrak{Qco} X)^{op}$$

with quasi-inverse $(-)^\circ U_\lambda$.

Proof. By Theorem 4.10 the localizing subcategory $\mathbb{K}_{m,Z}(\operatorname{Proj} X)$ is compactly generated in $\mathbb{K}_m(\operatorname{Proj} X)$. From the Neeman-Ravenel-Thomason localization theorem (Theorem 2.8) we deduce that

$$\mathbb{K}^c_{m,Z}(\operatorname{Proj} X) = \mathbb{K}_{m,Z}(\operatorname{Proj} X) \cap \mathbb{K}^c_m(\operatorname{Proj} X) \tag{9.4}$$

Taking the Spanier-Whitehead dual in $\mathbb{K}_m(\operatorname{Proj} X)$ commutes with restriction for compact objects (Proposition 6.12) so the following diagram commutes, up to natural equivalence

$$\begin{tikzcd}
\mathbb{K}^c_m(\operatorname{Proj} X) \arrow[r, "(-)|_U"] \arrow[d, "U(-)^\circ"', "\wr"] & \mathbb{K}^c_m(\operatorname{Proj} U) \arrow[d, "\wr"', "U(-)^\circ"] \\
\mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco} X)^{\mathrm{op}} \arrow[r, "(-)|_U"'] & \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco} U)^{\mathrm{op}}
\end{tikzcd}$$

The kernel of the top row is, by (9.4), the subcategory of compact objects $\mathbb{K}^c_{m,Z}(\operatorname{Proj} X)$, and the kernel of the bottom row is $\mathbb{D}^b_{\mathrm{coh},Z}(\mathfrak{Qco} X)^{\mathrm{op}}$ so we have the desired equivalence. $\square$

The local cohomology recollements for $\mathbb{D}(\mathfrak{Qco} X)$ and $\mathbb{K}_m(\operatorname{Proj} X)$, given in Proposition 9.1 and Theorem 9.3 above, are related. In fact, they fit into a kind of "exact sequence" of recollements, in which the "kernel" is a recollement involving the subcategory of acyclic complexes $\mathbb{K}_{m,\mathrm{ac},Z}(\operatorname{Proj} X)$ in $\mathbb{K}_{m,Z}(\operatorname{Proj} X)$.

Proposition 9.5. *There is a diagram in which each row and column is a recollement*

$$\begin{tikzcd}
\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} U) \arrow[r] \arrow[d] & \mathbb{K}_m(\operatorname{Proj} U) \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r] \arrow[d] & \mathbb{D}(\mathfrak{Qco} U) \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[d] \\
\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \arrow[u, shift left=2] \arrow[u, shift right=2] \arrow[r] \arrow[d] & \mathbb{K}_m(\operatorname{Proj} X) \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[u, shift left=2] \arrow[u, shift right=2] \arrow[r] \arrow[d] & \mathbb{D}(\mathfrak{Qco} X) \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[u, shift left=2] \arrow[u, shift right=2] \arrow[d] \\
\mathbb{K}_{m,\mathrm{ac},Z}(\operatorname{Proj} X) \arrow[u, shift left=2] \arrow[u, shift right=2] \arrow[r] & \mathbb{K}_{m,Z}(\operatorname{Proj} X) \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[u, shift left=2] \arrow[u, shift right=2] \arrow[r] & \mathbb{D}_Z(\mathfrak{Qco} X) \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[u, shift left=2] \arrow[u, shift right=2]
\end{tikzcd}$$

The diagram commutes if one restricts to arrows in the south and east directions.

Proof. Except for the left column and bottom row, these recollements are a consequence of Theorem 5.5 and the results of this section. With the notation of Chapter 5 the left adjoint U_λ of the canonical functor $U : \mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{D}(\mathfrak{Qco} X)$ calculates $\mathbb{K}$-flat resolutions and therefore commutes with restriction; taking right adjoints we find that the following diagram commutes up to natural equivalence (using the notation $\hat{\mathbb{R}} f_*$ for the right adjoint of restriction, introduced just prior to Lemma 6.8)

$$\begin{tikzcd}
\mathbb{K}_m(\operatorname{Proj} X) \arrow[r, "U"] & \mathbb{D}(\mathfrak{Qco} X) \\
\mathbb{K}_m(\operatorname{Proj} U) \arrow[r, "U"'] \arrow[u, "\hat{\mathbb{R}} f_*"] & \mathbb{D}(\mathfrak{Qco} U) \arrow[u, "\mathbb{R} f_*"']
\end{tikzcd}$$

By Theorem 9.3 the functor $\hat{\mathbb{R}} f_*$ is fully faithful, so we have a fully faithful functor $\hat{\mathbb{R}} f_* : \mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} U) \longrightarrow \mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ right adjoint to restriction. It preserves coproducts, and therefore has a right adjoint, because $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} U)$ is compactly generated (Theorem 5.5). By Lemma 2.3 and Lemma 2.6 we have a recollement

$$\begin{tikzcd}
\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} U) \arrow[r] & \mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r] & \mathbb{K}_{m,\mathrm{ac},Z}(\operatorname{Proj} X) \arrow[l, shift right=2] \arrow[l, shift left=2]
\end{tikzcd}$$

The functors U and U_λ restrict to an adjoint pair between $\mathbb{K}_{m,Z}(\operatorname{Proj} X)$ and $\mathbb{D}_Z(\mathfrak{Qco} X)$. Since U preserves coproducts and the category $\mathbb{K}_{m,Z}(\operatorname{Proj} X)$ is compactly generated, the restriction of U has a right adjoint. From Lemma 2.3 and Lemma 2.6 we conclude that there is a recollement

$$\begin{tikzcd}
\mathbb{K}_{m,\mathrm{ac},Z}(\operatorname{Proj} X) \arrow[r] & \mathbb{K}_{m,Z}(\operatorname{Proj} X) \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r] & \mathbb{D}_Z(\mathfrak{Qco} X) \arrow[l, shift right=2] \arrow[l, shift left=2]
\end{tikzcd}$$

which completes the proof. □

9.2 Characterizations of Smoothness

In this section we use our previous results to give a characterization of regular schemes, and show that $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ is an invariant of singularities, in the sense that it does not change upon restriction to an open subset containing all the singularities of X.

Setup. In this section X is a noetherian scheme and sheaves are defined over X by default.

The celebrated theorem of Serre [Ser56] and Auslander-Buchsbaum [AB56] states that a local noetherian ring is regular if and only if every finitely generated module is quasi-isomorphic to a bounded complex of finitely generated projectives. When this hypothesis fails, it is worthwhile to have an invariant that measures *how badly*. This invariant is called variously the *bounded stable derived category* or the *triangulated category of singularities*, and is defined to be the quotient

$$\mathbb{D}^b_{\mathrm{sg}}(X) = \mathbb{D}^b_{\mathrm{coh}}(\mathfrak{Qco}X)/\operatorname{Perf}(X)$$

of the bounded derived category of coherent sheaves, by the category of perfect complexes. It is a consequence of the classification of regular local rings that $\mathbb{D}^b_{\mathrm{sg}}(X)$ vanishes precisely when X is regular. We include a proof for the reader's convenience.

Proposition 9.6. *If X has finite Krull dimension then it is a regular scheme if and only if $\mathbb{D}^b_{\mathrm{sg}}(X) = 0$.*

Proof. Being a perfect complex is a local property, so the vanishing of $\mathbb{D}^b_{\mathrm{sg}}(X)$ is local. That is, if $\mathfrak{U} = \{U_0, \ldots, U_d\}$ is an affine open cover of X and $\mathbb{D}^b_{\mathrm{sg}}(U_i) = 0$ for each $0 \le i \le d$ then $\mathbb{D}^b_{\mathrm{sg}}(X) = 0$. Since being a regular scheme is also a local property, it suffices to prove the proposition when $X = Spec(A)$ for a noetherian ring A of finite Krull dimension.

Suppose that A is regular and let M be a finitely generated A-module. For each prime ideal $\mathfrak{p}$ the projective dimension of $M_{\mathfrak{p}}$ over $A_{\mathfrak{p}}$ is at most the Krull dimension of A, from which we deduce that $\operatorname{pd}_A(M) < \infty$. It follows that M is perfect as an object of $\mathbb{D}(A)$, whence any bounded complex of finitely generated modules is perfect and $\mathbb{D}^b_{\mathrm{sg}}(A) = 0$.

For the converse, we are given that $\mathbb{D}^b_{\mathrm{sg}}(A) = 0$ and we must prove that A is regular, for which it suffices to show that $A_{\mathfrak{m}}$ is regular for every maximal ideal $\mathfrak{m}$. Given $\mathfrak{m}$, the residue field $\kappa(\mathfrak{m}) = A_{\mathfrak{m}}/\mathfrak{m}A_{\mathfrak{m}} \cong A/\mathfrak{m}$ is a finitely generated A-module, so there is a finite resolution

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow \kappa(\mathfrak{m}) \longrightarrow 0$$

of $\kappa(\mathfrak{m})$ by finitely generated projective A-modules. Localizing at $\mathfrak{m}$ we have produced a finite projective resolution of $\kappa(\mathfrak{m})$ as an $A_{\mathfrak{m}}$-module, which implies that $A_{\mathfrak{m}}$ is regular. $\square$

From results earlier in this article and Krause's paper [Kra05] we know the infinite completion of the triangulated category of singularities and also of its opposite; combining Theorem 7.9 and [Kra05, Corollary 5.4] we have equivalences up to direct factors

$$\mathbb{D}^b_{\mathrm{sg}}(X)^{\mathrm{op}} \xrightarrow{\sim} \mathbb{K}^c_{m,\mathrm{ac}}(\operatorname{Proj} X) \tag{9.5}$$

$$\mathbb{D}^b_{\mathrm{sg}}(X) \xrightarrow{\sim} \mathbb{K}^c_{\mathrm{ac}}(\operatorname{Inj} X) \tag{9.6}$$

This leads to a new characterization of regular schemes, given below. One point we want to emphasize is that over regular schemes the mock homotopy category $\mathbb{K}_m(\operatorname{Proj} X)$ is canonically equivalent to the ordinary derived category.

Proposition 9.7. *The following conditions are equivalent*

(i) $\mathbb{D}^b_{\mathrm{sg}}(X) = 0$.

(ii) $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) = 0$.

(iii) Every complex of flat quasi-coherent sheaves is $\mathbb{K}$*-flat.*

(iv) The functor $\mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{D}(\mathfrak{Qco} X)$ *is an equivalence.*

Proof. The triangulated category $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ vanishes if and only if its subcategory of compact objects vanishes, which proves $(i) \Leftrightarrow (ii)$. By Theorem 5.5 we have a recollement

$$\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \leftleftarrows\!\!\!\rightarrow \mathbb{K}_m(\operatorname{Proj} X) \leftleftarrows\!\!\!\rightarrow \mathbb{D}(\mathfrak{Qco} X)$$

The rest of the proof follows by staring at this recollement. Firstly, we note that there is an equivalence $\mathbb{K}_m(\operatorname{Proj} X)/\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \xrightarrow{\sim} \mathbb{D}(\mathfrak{Qco} X)$ so the subcategory $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ is zero if and only if the canonical functor $\mathbb{K}_m(\operatorname{Proj} X) \longrightarrow \mathbb{D}(\mathfrak{Qco} X)$ is an equivalence, hence $(ii) \Leftrightarrow (iv)$. Next, observe that the subcategory $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ vanishes if and only if the orthogonal ${}^\perp\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ contains all complexes of flat quasi-coherent sheaves. This orthogonal is, by Proposition 5.2, the subcategory of $\mathbb{K}$-flat complexes, which implies $(ii) \Leftrightarrow (iii)$ and completes the proof. □

Remark 9.8. Suppose that X is regular and of finite Krull dimension, so that $\mathbb{D}^b_{\mathrm{sg}}(X) = 0$ by Proposition 9.6. Then by the previous proposition we have an equivalence

$$\mathbb{K}_m(\operatorname{Proj} X) \xrightarrow{\sim} \mathbb{D}(\mathfrak{Qco} X) \tag{9.7}$$

Since every complex of flat quasi-coherent sheaves is $\mathbb{K}$-flat, this is an equivalence of tensor triangulated categories; see also the proof of Proposition 6.4. If an equivalence identifies the tensor structures it must also identify the closed structures, so (9.7) is an equivalence of closed monoidal categories.

Combining several results of [Kra05] we have the injective analogue.

Proposition 9.9. *The following conditions are equivalent*

(i) $\mathbb{D}^b_{\mathrm{sg}}(X) = 0$.

(ii) $\mathbb{K}_{\mathrm{ac}}(\operatorname{Inj} X) = 0$.

(iii) Every complex of injective quasi-coherent sheaves is $\mathbb{K}$*-injective.*

(iv) The functor $\mathbb{K}(\operatorname{Inj} X) \longrightarrow \mathbb{D}(\mathfrak{Qco} X)$ *is an equivalence.*

Proof. The relevant results are [Kra05, Corollary 5.4] for $(i) \Leftrightarrow (ii)$ and [Kra05, Corollary 4.3] for $(ii) \Leftrightarrow (iv)$. The identification of the orthogonal $\mathbb{K}_{\mathrm{ac}}(\operatorname{Inj} X)^\perp$ with the subcategory of $\mathbb{K}$-injective complexes occurs in [Kra05, Corollary 3.9]. □

Specializing, we have the following characterization of regular rings.

Corollary 9.10. *Given a noetherian ring A of finite Krull dimension, the following are equivalent:*

(i) *A is regular.*

(ii) *Every complex of injective A-modules is $\mathbb{K}$-injective.*

(iii) *Every complex of projective A-modules is $\mathbb{K}$-projective.*

(iv) *Every complex of flat A-modules is $\mathbb{K}$-flat.*

Proof. To prove $(i) \Leftrightarrow (ii) \Leftrightarrow (iv)$ one combines Propositions 9.6, 9.7 and 9.9. To see that (iii) is equivalent to the other conditions, consider the recollement of Theorem 5.15

$$\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A) \rightleftarrows \mathbb{K}(\operatorname{Proj} A) \rightleftarrows \mathbb{D}(A)$$

As in Proposition 9.7 we argue that $\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ vanishes precisely when A is a regular ring. Moreover, the left orthogonal ${}^{\perp}\mathbb{K}_{\mathrm{ac}}(\operatorname{Proj} A)$ is the subcategory of $\mathbb{K}$-projective complexes (Corollary 5.14) from which we deduce $(i) \Leftrightarrow (iii)$. □

We can now prove that $\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ is an invariant of the singularities of our scheme. This gives the unbounded analogue of a result of Orlov for the triangulated category of singularities [Orl04, Proposition 1.14], and should be compared to [Kra05, Corollary 6.10], which gives the corresponding statement for the homotopy category of acyclic complexes of injective sheaves.

Proposition 9.11. *If $U \subseteq X$ is an open subset containing every singularity of X then the restriction functor*

$$(-)|_U : \mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \longrightarrow \mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} U) \tag{9.8}$$

is an equivalence of triangulated categories.

Proof. By Proposition 9.5 the restriction functor fits into a recollement (setting $Z = X \setminus U$)

$$\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} U) \rightleftarrows \mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) \rightleftarrows \mathbb{K}_{m,\mathrm{ac},Z}(\operatorname{Proj} X)$$

In particular the functor (9.8) induces an equivalence

$$\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X) / \mathbb{K}_{m,\mathrm{ac},Z}(\operatorname{Proj} X) \xrightarrow{\sim} \mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} U)$$

To prove that (9.8) is an equivalence, we show that the kernel $\mathbb{K}_{m,\mathrm{ac},Z}(\operatorname{Proj} X)$ vanishes whenever U contains all the singularities of X. Intuitively, $\mathbb{K}_{m,\mathrm{ac},Z}(\operatorname{Proj} X)$ is the invariant of singularities of X contained in Z, which should vanish under the stated conditions.

Let $\mathscr{F}$ be an acyclic complex of flat quasi-coherent sheaves on X that is mock supported on Z, so that $\mathscr{F}|_U$ is $\mathbb{K}$-flat. For any point $x \notin U$ the local ring $\mathcal{O}_{X,x}$ is regular, so $\mathscr{F}_x$ is $\mathbb{K}$-flat by Corollary 9.10(iv). We deduce that $\mathscr{F}$ is $\mathbb{K}$-flat on stalks for every point of X, whence $\mathscr{F}$ is $\mathbb{K}$-flat globally and thus zero in $\mathbb{K}_m(\operatorname{Proj} X)$. This proves that $\mathbb{K}_{m,\mathrm{ac},Z}(\operatorname{Proj} X)$ is the zero category, as required. □

Appendix A

Flat Covers of Complexes

In this appendix we prove that for any noetherian scheme X the inclusion

$$J : \mathbb{K}(\operatorname{Flat} X) \longrightarrow \mathbb{K}(\mathfrak{Qco} X) \tag{A.1}$$

has a right adjoint. This adjoint will be used in Appendix B to introduce a closed monoidal structure on $\mathbb{K}(\operatorname{Flat} X)$ and $\mathbb{K}(\operatorname{Inj} X)$. The proof consists of generalizing an argument of Neeman for rings [Nee06c] using some ideas of Enochs and Estrada from [EE05b]. Let us first outline the proof in the affine case and then explain the argument that will lead to our generalization. Given a ring A there exists by [Nee06c, Remark 3.2] a recollement (using the notation of Remark 3.5)

$$\mathbb{E}(A) \rightleftarrows \mathbb{K}(\operatorname{Flat} A) \rightleftarrows \mathbb{K}(\operatorname{Proj} A) \tag{A.2}$$

The inclusion $J : \mathbb{K}(\operatorname{Flat} A) \longrightarrow \mathbb{K}(A)$ can be "decomposed" into two pieces, the respective inclusions of the two subcategories in the recollement

$$\mathbb{E}(A) \longrightarrow \mathbb{K}(A), \qquad \mathbb{K}(\operatorname{Proj} A) \longrightarrow \mathbb{K}(A) \tag{A.3}$$

To prove that J has a right adjoint one constructs a right adjoint for each piece in (A.3). The second inclusion has a right adjoint by Brown-Neeman representability, as $\mathbb{K}(\operatorname{Proj} A)$ is well generated [Nee06a, Theorem 4.8], and for first inclusion Neeman uses the following result on constructing adjoints from precovers.

Definition A.1. Let $\mathcal{T}$ be a category and $\mathcal{S}$ a full subcategory. A morphism $s \longrightarrow t$ is called an *$\mathcal{S}$-precover* of t if $s \in \mathcal{S}$ and every morphism $s' \longrightarrow t$ with $s' \in \mathcal{S}$ factors (not necessarily uniquely) through $s \longrightarrow t$. An important example is the notion of a *flat precover* of a module, which we have already seen in Definition 2.32.

Proposition A.2. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S} \subseteq \mathcal{T}$ a thick triangulated subcategory. Assume that*

(i) Every object $t \in \mathcal{T}$ admits an $\mathcal{S}$-precover.

(ii) Every idempotent in $\mathcal{T}$ splits.

Then the inclusion $F : \mathcal{S} \longrightarrow \mathcal{T}$ has a right adjoint.

Proof. See [Nee06c, Proposition 1.4]. □

In our case, we learn that to construct a right adjoint of $\mathbb{E}(A) \longrightarrow \mathbb{K}(A)$ it is enough to construct $\mathbb{E}(A)$-precovers. To do this, Neeman introduces an auxiliary (noncommutative) ring, denoted here by $\mathbb{N}(A)$, such that the category of complexes of A-modules embeds in the category of $\mathbb{N}(A)$-modules. Flat precovers over $\mathbb{N}(A)$ give $\mathbb{E}(A)$-precovers of complexes, so both pieces in (A.3) have a right adjoint and thus so does J.

It is worth mentioning an earlier result of Enochs and Rozas [EGR98]. They call the complexes in $\mathbb{E}(A)$ *flat* and prove that over a commutative noetherian ring of finite Krull dimension, any complex of modules has a "flat precover". The difference is that Neeman works in the homotopy category (and with arbitrary noncommutative rings) while Enochs and Rozas work in the category of complexes.

We aim to prove that, for a noetherian scheme X, the inclusion J of (A.1) has a right adjoint. The proof arises by generalizing each step in the above argument for rings, and the first ingredient is the following localization sequence (Theorem 3.16)

$$\mathbb{E}(X) \rightleftarrows \mathbb{K}(\operatorname{Flat} X) \rightleftarrows \mathbb{K}_m(\operatorname{Proj} X)$$

by which we reduce to constructing a right adjoint for the inclusion $\mathbb{E}(X) \longrightarrow \mathbb{K}(\mathfrak{Qco}X)$. From Proposition A.2 we learn that it is enough to construct $\mathbb{E}(X)$-precovers. Following [EGR98, §2] we think of quasi-coherent sheaves on X as modules over a representation R in the category of rings of a certain quiver Q. We introduce an auxiliary presheaf $\mathbb{N}(R)$ of (noncommutative) rings, and observe that the category of complexes of R-modules embeds in the category of $\mathbb{N}(R)$-modules. Flat precovers in the latter category give rise to $\mathbb{E}(R)$-precovers in the former (Lemma A.11) and using the correspondence between R-modules and sheaves, this will yield the desired $\mathbb{E}(X)$-precovers. At this point it is a short step to Theorem A.13 where we prove that J has a right adjoint.

Setup. In this section rings may be noncommutative, and X denotes a fixed scheme.

Let us define Neeman's auxiliary ring. The following construction and its properties are described in [Nee06c] but we repeat the definitions here for the reader's convenience.

Definition A.3. [Nee06c, Notation 2.3] Let A be a ring, which, by the conventions of this section, may be noncommutative. We study the ring constructed from A and the quiver

$$\cdots \xrightarrow[\partial^{i-2}]{} \bullet \xrightarrow[\partial^{i-1}]{} \bullet \xrightarrow[\partial^{i}]{} \bullet \xrightarrow[\partial^{i+1}]{} \cdots$$

with the relation $\partial^{i+1}\partial^i = 0$. To be precise, we introduce the set $S = \{\partial^i, e^i\}_{i\in\mathbb{Z}}$ and the graded ring $A\langle S\rangle$ (the free noncommutative A-algebra on S) which is the free A-module on the set of sequences in S (including the empty sequence). There is a canonical ring morphism $A \longrightarrow A\langle S\rangle$. Consider the two sided ideal I of $A\langle S\rangle$ generated by the following relations

$$\begin{aligned} e^i e^j &= 0 \quad \text{ if } i \neq j \\ e^i e^i &= e^i \\ \partial^i \partial^j &= 0 \\ e^{i+1}\partial^i &= \partial^i e^i = \partial^i \\ e^j \partial^i &= 0 \text{ unless } j = i+1 \\ \partial^i e^j &= 0 \text{ unless } j = i \end{aligned}$$

Write $\mathbb{N}(A)$ for the ring $A\langle S\rangle/I$. It is not difficult to check that $\mathbb{N}(A)$ is free as an A-module on the basis $\{1, \partial^i, e^i\}_{i\in\mathbb{Z}}$. A morphism of *commutative* rings $A \longrightarrow B$ induces a morphism of rings $A\langle S\rangle \longrightarrow B\langle S\rangle$ and thus $\mathbb{N}(A) \longrightarrow \mathbb{N}(B)$, and this makes the construction $\mathbb{N}(-)$ into a functor from commutative rings to rings.

Definition A.4. Let A be a ring, Z a complex of A-modules

$$\cdots \longrightarrow Z^{i-1} \longrightarrow Z^i \longrightarrow Z^{i+1} \longrightarrow \cdots$$

and denote by $T(Z)$ the $\mathbb{N}(A) = A\langle S\rangle/I$-module, which as an A-module is the coproduct

$$T(Z) = \bigoplus_{i\in\mathbb{Z}} Z^i$$

with the action of $\mathbb{N}(A)$ defined by the action of the generators ∂^i, e^i as

$$\begin{aligned} \partial^i \cdot (\ldots, z^{i-1}, z^i, z^{i+1}, \ldots) &= (\ldots, 0, 0, \partial^i(z^i), 0, \ldots) \\ e^i \cdot (\ldots, z^{i-1}, z^i, z^{i+1}, \ldots) &= (\ldots, 0, z^i, 0, \ldots) \end{aligned}$$

Given a morphism $\phi : Z \longrightarrow Q$ of complexes of A-modules, $T(\phi) = \oplus_i \phi^i$ is a morphism of $\mathbb{N}(A)$-modules, so this defines an additive functor $T : \mathbb{C}(A) \longrightarrow \mathbb{N}(A)\mathbf{Mod}$. One checks that T is fully faithful. In the other direction, let a $\mathbb{N}(A)$-module M be given, and let $T_\rho(M)$ denote the following complex of A-modules

$$\cdots \longrightarrow e^{i-1}M \xrightarrow{\partial^{i-1}} e^i M \xrightarrow{\partial^i} e^{i+1}M \longrightarrow \cdots$$

A morphism $M \longrightarrow N$ of $\mathbb{N}(A)$-modules restricts to a sequence of maps $e^iM \longrightarrow e^iN$, defining a morphism of complexes $T_\rho(M) \longrightarrow T_\rho(N)$. This defines an additive functor $T_\rho : \mathbb{N}(A)\mathbf{Mod} \longrightarrow \mathbb{C}(A)$.

Given an $\mathbb{N}(A)$-module M the inclusions $e^iM \longrightarrow M$ give a morphism of $\mathbb{N}(A)$-modules $\varepsilon : TT_\rho(M) \longrightarrow M$ natural in M. This is the counit of an adjunction, with T left adjoint to T_ρ. In [Nee06c, Proposition 2.8] it is shown that T_ρ sends flat $\mathbb{N}(A)$-modules to complexes in $\mathbb{E}(A)$, and in the reverse direction T sends complexes from $\mathbb{E}(A)$ to flat $\mathbb{N}(A)$-modules (the notation $\mathbb{E}(A)$ was introduced in Remark 3.5, and agrees with Neeman's $\mathcal{S}$).

Lemma A.5. *Let $A \longrightarrow B$ be a flat morphism of commutative rings. The induced ring morphism $\mathbb{N}(A) \longrightarrow \mathbb{N}(B)$ makes $\mathbb{N}(B)$ flat as both a left and right $\mathbb{N}(A)$-module.*

Proof. The ring $\mathbb{N}(B) = B\langle S\rangle/I$ is isomorphic as a left and right $\mathbb{N}(A)$-module to the algebra $B \otimes_A \mathbb{N}(A)$, which is a module via the ring morphism $\mathbb{N}(A) \longrightarrow B \otimes_A \mathbb{N}(A)$ sending a to $1 \otimes a$. Given this observation, the claim is straightforward to check. $\square$

It is known that flat covers exist in the category of quasi-coherent sheaves on a scheme [EE05b]. The proof works by replacing the category of quasi-coherent sheaves by a category of modules over a presheaf of commutative rings, defined on a certain quiver Q. Let us recall this construction. Let Q be a quiver and $\mathscr{C}(Q)$ the path category of Q. Denoting by **Rng** the category of (noncommutative) rings, a contravariant functor $\mathscr{C}(Q) \longrightarrow \mathbf{Rng}$ is a *presheaf of rings* on Q. In [EE05b] a dual definition is used, but presheaves are more natural in the present context and the distinction is trivial, in any case.

Let R be a presheaf of rings on Q. A *left module* over R is a presheaf M of abelian groups on Q with the property that, for every edge $a : v \longrightarrow w$, the morphism $M(w) \longrightarrow M(v)$ is a morphism of $R(w)$-modules. The category $R\mathbf{Mod}$ of left modules over R is Grothendieck abelian. Similarly we define the Grothendieck abelian category $\mathbf{Mod}R$ of right modules over R. Given a left module M over R and an edge $a : v \longrightarrow w$ we have a morphism of $R(v)$-modules natural in M

$$\begin{gathered} R(v) \otimes_{R(w)} M(w) \longrightarrow M(v) \\ r \otimes m \mapsto r \cdot M(a)(m) \end{gathered}$$

We say that M is *quasi-coherent* if this is an isomorphism for every edge $a : v \longrightarrow w$. The quasi-coherent modules define an abelian subcategory $\mathfrak{Qco}(R)$ of $R\mathbf{Mod}$, provided that R is *flat*: this means that for each edge $a : v \longrightarrow w$ the ring morphism $R(w) \longrightarrow R(v)$ makes $R(v)$ a flat right $R(w)$-module. There is a natural tensor product for modules over R, that we can use to define *flat* right and left R-modules; see [EE05b]. In this section, modules over a presheaf of rings are left modules unless indicated otherwise.

Combining the following results of [EE05b] gives flat covers for quasi-coherent sheaves.

Proposition A.6. *There exists a quiver Q and a flat presheaf of commutative rings R on Q, such that the category of quasi-coherent R-modules is equivalent to $\mathfrak{Qco}(X)$. Moreover, this equivalence preserves flatness in both directions.*

Proof. See [EE05b, §2]. $\square$

Theorem A.7 (Enochs,Estrada)**.** *Let Q be a quiver and R a flat presheaf of rings defined on Q. The category of quasi-coherent R-modules $\mathscr{C}$ admits flat precovers.*

Proof. See [EE05b, Theorem 4.1]. $\square$

For the remainder of this section let Q be a quiver and R a presheaf of commutative rings on Q. Applying the functor $\mathbb{N}(-)$ we have a presheaf $\mathbb{N}(R)$ of rings on Q, defined by $\mathbb{N}(R)(v) = \mathbb{N}(R(v))$, which is flat if R is flat, by Lemma A.5. We want to construct the $\mathbb{E}(R)$-precover (defined below) of a complex of R-modules by "packing" the complex into a single module over $\mathbb{N}(R)$ and taking the flat precover of this $\mathbb{N}(R)$-module. To do this we generalize Definition A.4, and we hope the reader will bear with us through some technical detail. For each $v \in Q$ we have an adjoint pair, described by Definition A.4 when $A = R(v)$

$$\mathbb{C}(R(v)\mathbf{Mod}) \underset{T_{v,\rho}}{\overset{T_v}{\rightleftarrows}} \mathbb{N}(R)(v)\mathbf{Mod} \tag{A.4}$$

Let a complex Z of R-modules be given. If we fix $v \in Q$ then we have a complex of $R(v)$-modules $Z(v)$ and thus an $\mathbb{N}(R)(v)$-module $T(Z)(v) = T_v(Z(v))$. This defines an $\mathbb{N}(R)$-module $T(Z)$ and a fully faithful additive functor

$$T : \mathbb{C}(R\mathbf{Mod}) \longrightarrow \mathbb{N}(R)\mathbf{Mod}$$

realizing complexes of R-modules as modules over the presheaf of rings $\mathbb{N}(R)$.

It remains to define the right adjoint, which unpacks an $\mathbb{N}(R)$-module to give a complex. Given an $\mathbb{N}(R)$-module M we have for an integer $i \in \mathbb{Z}$ and vertex v an $R(v)$-module $T_{v,\rho}(M(v))^i$, and we define a complex $T_\rho(M)$ of R-modules by $T_\rho(M)^i(v) = T_{v,\rho}(M(v))^i$. With a little checking this defines an additive functor

$$T_\rho : \mathbb{N}(R)\mathbf{Mod} \longrightarrow \mathbb{C}(R\mathbf{Mod})$$

Define a natural transformation $\varepsilon : TT_\rho \longrightarrow 1$ by setting $\varepsilon_M(v) = \varepsilon_{v,M(v)}$ for an $\mathbb{N}(R)$-module M, where $\varepsilon_v : T_v T_{v,\rho} \longrightarrow 1$ is the counit for the adjunction (A.4). One checks that ε is the counit of an adjunction, with T left adjoint to T_ρ

$$\mathbb{C}(R\mathbf{Mod}) \underset{T_{\rho}}{\overset{T}{\rightleftarrows}} \mathbb{N}(R)\mathbf{Mod} \tag{A.5}$$

Next we show that the adjoint functors T and T_ρ interchange flat $\mathbb{N}(R)$-modules and acyclic complexes of flat R-modules with flat kernels.

Lemma A.8. *If Z is an acyclic complex of flat R-modules with flat kernels then $T(Z)$ is a flat $\mathbb{N}(R)$-module. In the other direction, if M is a flat $\mathbb{N}(R)$-module then $T_\rho(M)$ is an acyclic complex of flat R-modules with flat kernels.*

Proof. Given an acyclic complex Z of flat R-modules with flat kernels, the complex $Z(v)$ of $R(v)$-modules belongs to $\mathbb{E}(R(v))$, so $T(Z(v))$ is flat as an $\mathbb{N}(R)(v)$-module by [Nee06c, Proposition 2.8]. It follows that $T(Z)$ is a flat $\mathbb{N}(R)$-module. For the second claim, let M be a flat $\mathbb{N}(R)$-module. Then $M(v)$ is flat for every vertex v, from which we deduce that $T_{v,\rho}(M(v))$ belongs to $\mathbb{E}(R(v))$. We deduce that $T_\rho(M)$ is an acyclic complex of flat $R(v)$-modules with flat kernels. □

We are interested in quasi-coherent sheaves and thus, quasi-coherent R-modules. An important property of the functors T and T_ρ is that they both preserve quasi-coherence; before giving the proof, we need a technical lemma.

Lemma A.9. *Let $a : v \longrightarrow w$ be an edge, and S a left $\mathbb{N}(R)(w)$-module. The map*

$$R(v) \otimes_{R(w)} S \longrightarrow \mathbb{N}(R)(v) \otimes_{\mathbb{N}(R)(w)} S$$
$$\lambda \otimes x \mapsto \lambda \otimes x$$

is an isomorphism of left $R(v)$-modules.

Proof. Such a map clearly exists. Its inverse is the morphism induced out of the tensor product $\mathbb{N}(R)(v) \otimes_{\mathbb{N}(R)(w)} S$ by the following $\mathbb{N}(R)(w)$-bilinear map

$$\mathbb{N}(R(v)) \times S \longrightarrow R(v) \otimes_{R(w)} S$$
$$\left(\lambda + \sum_i \mu_i \partial^i + \sum_j \tau_j e^j, x\right) \mapsto \lambda \otimes x + \sum_i (\mu_i \otimes \partial^i x) + \sum_j (\tau_j \otimes e^j x)$$

where we use the fact that $\mathbb{N}(R(v))$ is free as an $\mathbb{R}(v)$-module on the set $\{1, \partial^i, e^i\}_{i \in \mathbb{Z}}$ to write an arbitrary element in terms of coefficients $\lambda, \mu_i, \tau_j \in R(v)$. $\square$

The following result shows that the constructions T and T_ρ preserve quasi-coherence.

Lemma A.10. *Suppose that R is flat. Then*

(i) If Z is a complex of quasi-coherent R-modules then $T(Z)$ is a quasi-coherent $\mathbb{N}(R)$-module.

(ii) If M is a quasi-coherent $\mathbb{N}(R)$-module then $T_\rho(M)$ is a complex of quasi-coherent R-modules.

Proof. (i) Let Z be a complex of quasi-coherent R-modules, and $a : v \longrightarrow w$ an edge. We have to show that the canonical morphism $\mathbb{N}(R(v)) \otimes_{\mathbb{N}(R(w))} T(Z)(w) \longrightarrow T(Z)(v)$ is an isomorphism. Using Lemma A.9 this reduces to showing that the map

$$R(v) \otimes_{R(w)} T(Z)(w) \longrightarrow T(Z)(v)$$
$$r \otimes m \mapsto r \cdot T(Z)(a)(m)$$

is an isomorphism. As an $R(w)$-module we have $T(Z)(w) = \oplus_{i \in \mathbb{Z}} Z^i(w)$, and similarly for v, and moreover the tensor product commutes with coproducts, so it is enough to show that the map $R(v) \otimes_{R(w)} Z^i(w) \longrightarrow Z^i(v)$ is an isomorphism for every $i \in \mathbb{Z}$. But this is known, because each Z^i was assumed quasi-coherent.

(ii) Given a quasi-coherent $\mathbb{N}(R)$-module M and an edge $a : v \longrightarrow w$, consider the following commutative diagram for $I \in \mathbb{Z}$

$$\begin{tikzcd}
R(v) \otimes_{R(w)} e^I M(w) \arrow[r, "h"] \arrow[d, "t"'] & e^I M(v) \arrow[dd] \\
R(v) \otimes_{R(w)} M(w) \arrow[d, "\wr"'] & \\
\mathbb{N}(R(v)) \otimes_{\mathbb{N}(R(w))} M(w) \arrow[r, "H"'] & M(v)
\end{tikzcd} \tag{A.6}$$

where $e^I M(w) \longrightarrow M(w)$ is the inclusion, and we again use Lemma A.9. To show that $T_\rho(M)$ is a complex of quasi-coherent R-modules, we have to show that h is an isomorphism. By assumption the bottom row is an isomorphism, and since R is flat the map marked t is injective. From this we infer that h is injective.

To see that h is surjective, let $x \in M(v)$ be given. Because M is quasi-coherent, we can write $x = \sum_i H(X_i \otimes m_i)$ for some $X_i \in \mathbb{N}(R(v))$ and $m_i \in M(w)$. Therefore $e^I x = \sum_i H(e^I X_i \otimes m_i)$ for any $I \in \mathbb{Z}$. For each i we can write X_i using the canonical basis in $\mathbb{N}(R(v))$

$$X_i = \lambda_i + \sum_t \mu_{i,t}\partial^t + \sum_s \tau_{i,s} e^s$$

and one calculates

$$e^I x = \sum_i h(\lambda_i \otimes e^I m_i + \mu_{i,I-1} \otimes e^I \partial^{I-1} m_i + \tau_{i,I} \otimes e^I m_i)$$

which shows that h is surjective, hence an isomorphism. □

Let R be a flat presheaf of commutative rings, so that $\mathfrak{Qco}(R)$ is an abelian category. Denote by $\mathbb{K}(\mathrm{Flat}\, R) \subseteq \mathbb{K}(\mathfrak{Qco} R)$ the homotopy category of flat quasi-coherent R-modules and by $\mathbb{E}(R)$ the full subcategory of acyclic complexes with flat kernels in $\mathbb{K}(\mathrm{Flat}\, R)$.

Lemma A.11. *Let Q be a quiver and R a flat presheaf of commutative rings. Every complex of quasi-coherent R-modules has an $\mathbb{E}(R)$-precover in the category $\mathbb{K}(\mathfrak{Qco} R)$.*

Proof. To be clear: we are claiming that for any complex Z of quasi-coherent R-modules there is a complex E in $\mathbb{E}(R)$ and morphism of complexes $E \longrightarrow Z$ with the property that any morphism of complexes $E' \longrightarrow Z$ with E' in $\mathbb{E}(R)$ factors (not necessarily uniquely) via $E \longrightarrow Z$ in $\mathbb{K}(\mathfrak{Qco} R)$. In fact, the factorization will happen on the level of complexes.

Let Z be a complex of quasi-coherent R-modules. Packing the complex into a single module will produce, by Lemma A.10, a quasi-coherent $\mathbb{N}(R)$-module $T(Z)$. We know from Theorem A.7 that modules over such a presheaf of rings have flat precovers; let $F \longrightarrow T(Z)$ such a precover, in the category of quasi-coherent $\mathbb{N}(R)$-modules.

Applying T_ρ to unpack our modules into complexes, and using the equivalence $T_\rho T \cong 1$, we have a morphism $T_\rho(F) \longrightarrow Z$ in the category of complexes of R-modules. Together, Lemma A.8 and Lemma A.10 tell us that $T_\rho(F)$ is a complex of flat quasi-coherent R-modules with flat kernels, and one checks that this is the desired $\mathbb{E}(R)$-precover. □

Proposition A.12. *The inclusion $\mathbb{E}(X) \longrightarrow \mathbb{K}(\mathfrak{Qco} X)$ has a right adjoint.*

Proof. By Proposition A.6 there is a quiver Q and a flat presheaf of commutative rings R on Q, together with an equivalence of categories

$$\mathfrak{Qco}(R) \xrightarrow{\sim} \mathfrak{Qco}(X)$$

identifying the subcategories of flat objects on both sides. We will apply Proposition A.2 to deduce a right adjoint for the inclusion $\mathbb{E}(X) \longrightarrow \mathbb{K}(\mathfrak{Qco} X)$. The category $\mathbb{K}(\mathfrak{Qco} X)$ has coproducts, so any idempotent splits [Nee01b, Proposition 1.6.8], and we already know that $\mathbb{E}(X) \subseteq \mathbb{K}(\mathfrak{Qco}X)$ is thick, so it remains to prove that every object in $\mathbb{K}(\mathfrak{Qco}X)$ has an $\mathbb{E}(X)$-precover. If we identify, up to equivalence, the category $\mathbb{K}(\mathfrak{Qco}X)$ with the category $\mathbb{K}(\mathfrak{Qco}R)$ and $\mathbb{E}(X)$ with $\mathbb{E}(R)$, this follows from Lemma A.11. □

Theorem A.13. *If X is noetherian then the inclusion*

$$J : \mathbb{K}(\operatorname{Flat} X) \longrightarrow \mathbb{K}(\mathfrak{Qco}X)$$

has a right adjoint.

Proof. We prove that J has a right adjoint by constructing, for every complex $\mathscr{F}$ of quasi-coherent sheaves, a triangle in $\mathbb{K}(\mathfrak{Qco}X)$ with $\mathscr{C}$ in $\mathbb{K}(\operatorname{Flat} X)$ and $\mathscr{T}$ in $\mathbb{K}(\operatorname{Flat} X)^{\perp}$

$$\mathscr{C} \longrightarrow \mathscr{F} \longrightarrow \mathscr{T} \longrightarrow \Sigma\mathscr{C} \tag{A.7}$$

This is done in two stages: first we construct the triangle in $\mathbb{K}(\mathfrak{Qco}X)$ modulo $\mathbb{E}(X)$, then we lift the triangle to $\mathbb{K}(\mathfrak{Qco}X)$. If T denotes the quotient $\mathbb{K}(\mathfrak{Qco}X) \longrightarrow \mathbb{K}(\mathfrak{Qco}X)/\mathbb{E}(X)$ then from Lemma 2.3 and Proposition A.12 we deduce a localization sequence

$$\begin{tikzcd} \mathbb{E}(X) \arrow[r, shift left] & \mathbb{K}(\mathfrak{Qco}X) \arrow[l, shift left] \arrow[r, shift left, "T"] & \mathbb{K}(\mathfrak{Qco}X)/\mathbb{E}(X) \arrow[l, shift left] \end{tikzcd} \tag{A.8}$$

Recall the following fact: given a triangulated category $\mathcal{T}$ and triangulated subcategories $\mathcal{S} \subseteq \mathcal{Q} \subseteq \mathcal{T}$ the induced triangulated functor $\mathcal{Q}/\mathcal{S} \longrightarrow \mathcal{T}/\mathcal{S}$ is fully faithful. In our specific case, the inclusion $J : \mathbb{K}(\operatorname{Flat} X) \longrightarrow \mathbb{K}(\mathfrak{Qco}X)$ induces a fully faithful triangulated functor M making the following diagram commute

$$\begin{tikzcd} \mathbb{K}(\mathfrak{Qco}X) \arrow[r, "T"] & \mathbb{K}(\mathfrak{Qco}X)/\mathbb{E}(X) \\ \mathbb{K}(\operatorname{Flat} X) \arrow[u, "J"] \arrow[r, "Q"'] & \mathbb{K}_m(\operatorname{Proj} X) \arrow[u, "M"'] \end{tikzcd}$$

Because M preserves coproducts and $\mathbb{K}_m(\operatorname{Proj} X)$ is compactly generated (Theorem 4.10) M has a right adjoint M_ρ. Thus, for our given complex $\mathscr{F}$ of quasi-coherent sheaves, we can find a triangle in $\mathbb{K}(\mathfrak{Qco}X)/\mathbb{E}(X)$

$$MM_\rho(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow \mathscr{M}' \longrightarrow \Sigma MM_\rho(\mathscr{X})$$

with $\operatorname{Hom}_{\mathbb{K}(\mathfrak{Qco}X)/\mathbb{E}(X)}(\mathscr{A}, \mathscr{M}') = 0$ for every $\mathscr{A}$ in $\mathbb{K}(\operatorname{Flat} X)$. This is the triangle (A.7) that we are looking for, modulo $\mathbb{E}(X)$. It remains to lift the triangle to $\mathbb{K}(\mathfrak{Qco}X)$.

Write the counit $MM_\rho(\mathscr{F}) \longrightarrow \mathscr{F}$ as a composite $T(g)T(f)^{-1}$ for a pair of morphisms $g : \mathscr{P} \longrightarrow \mathscr{F}$ and $f : \mathscr{P} \longrightarrow MM_\rho(\mathscr{F})$ in $\mathbb{K}(\mathfrak{Qco}X)$ with f having mapping cone in $\mathbb{E}(X)$. Since $\mathscr{P}$ can be written as the mapping cone on a morphism between $MM_\rho(\mathscr{F})$ and an object of $\mathbb{E}(X)$, both of which are complexes of flat quasi-coherent sheaves, we can assume that $\mathscr{P}$ is a complex of flat quasi-coherent sheaves. Extending g to a triangle in $\mathbb{K}(\mathfrak{Qco}X)$, we have

$$\mathscr{P} \xrightarrow{g} \mathscr{F} \longrightarrow \mathscr{M} \longrightarrow \Sigma\mathscr{P} \tag{A.9}$$

with $\mathrm{Hom}_{\mathbb{K}(\mathfrak{Qco}X)/\mathbb{E}(X)}(\mathscr{A},\mathscr{M}) = 0$ for every $\mathscr{A}$ in $\mathbb{K}(\mathrm{Flat}\, X)$, because $\mathscr{M}$ and $\mathscr{M}'$ are isomorphic in $\mathbb{K}(\mathfrak{Qco}X)/\mathbb{E}(X)$. As $\mathbb{E}(X)$ is a Bousfield subcategory of $\mathbb{K}(\mathfrak{Qco}X)$ we can find a triangle in $\mathbb{K}(\mathfrak{Qco}X)$ with $\mathscr{R}$ in $\mathbb{E}(X)$ and $\mathscr{T}$ in the orthogonal $\mathbb{E}(X)^{\perp}$

$$\mathscr{R} \longrightarrow \mathscr{M} \longrightarrow \mathscr{T} \longrightarrow \Sigma\mathscr{R} \tag{A.10}$$

Since $\mathscr{R}$ vanishes in the quotient $\mathbb{K}(\mathfrak{Qco}X)/\mathbb{E}(X)$ we deduce that for $\mathscr{A}$ in $\mathbb{K}(\mathrm{Flat}\, X)$

$$\mathrm{Hom}_{\mathbb{K}(\mathfrak{Qco}X)}(\mathscr{A},\mathscr{T}) \cong \mathrm{Hom}_{\mathbb{K}(\mathfrak{Qco}X)/\mathbb{E}(X)}(\mathscr{A},\mathscr{T}) \cong \mathrm{Hom}_{\mathbb{K}(\mathfrak{Qco}X)/\mathbb{E}(X)}(\mathscr{A},\mathscr{M}) = 0$$

where the first isomorphism is a standard property of Verdier quotients. We have shown that $\mathscr{T}$ belongs to $\mathbb{K}(\mathrm{Flat}\, X)^{\perp}$ with the orthogonal taken in $\mathbb{K}(\mathfrak{Qco}X)$. From (A.9), (A.10) and the octahedral axiom, we deduce two new triangles in $\mathbb{K}(\mathfrak{Qco}X)$

$$\mathscr{C} \longrightarrow \mathscr{F} \longrightarrow \mathscr{T} \longrightarrow \Sigma\mathscr{C}$$
$$\mathscr{P} \longrightarrow \mathscr{C} \longrightarrow \mathscr{R} \longrightarrow \Sigma\mathscr{P}$$

Since both $\mathscr{P}$ and $\mathscr{R}$ belong to $\mathbb{K}(\mathrm{Flat}\, X)$, we can assume that $\mathscr{C}$ is also a complex of flat quasi-coherent sheaves. Hence the first triangle of this pair has $\mathscr{C}$ in $\mathbb{K}(\mathrm{Flat}\, X)$ and $\mathscr{T}$ in $\mathbb{K}(\mathrm{Flat}\, X)^{\perp}$ and the proof is complete. ☐

Next we give a counterexample to show that certain adjoints do not exist in general.

Corollary A.14. *If X is noetherian and either of the canonical functors*

$$\mathbb{K}(\mathrm{Flat}\, X) \longrightarrow \mathbb{D}(\mathfrak{Qco}X), \quad \mathbb{K}(\mathrm{Flat}\, X) \longrightarrow \mathbb{K}_m(\mathrm{Proj}\, X) \tag{A.11}$$

have left adjoints then products are exact in $\mathfrak{Qco}(X)$.

Proof. By the theorem the inclusion $J : \mathbb{K}(\mathrm{Flat}\, X) \longrightarrow \mathbb{K}(\mathfrak{Qco}X)$ has a right adjoint, so for every complex $\mathscr{F}$ of quasi-coherent sheaves there is a triangle in $\mathbb{K}(\mathfrak{Qco}X)$

$$JJ_{\rho}(\mathscr{F}) \longrightarrow \mathscr{F} \longrightarrow \mathscr{T} \longrightarrow \Sigma JJ_{\rho}(\mathscr{F})$$

with $\mathscr{T}$ belonging to $\mathbb{K}(\mathrm{Flat}\, X)^{\perp}$ and therefore acyclic (every complex has a $\mathbb{K}$-flat resolution by flat quasi-coherent sheaves). Applying the quotient $q : \mathbb{K}(\mathfrak{Qco}X) \longrightarrow \mathbb{D}(\mathfrak{Qco}X)$ to this triangle we deduce a natural equivalence

$$q \xrightarrow{\sim} qJJ_{\rho}$$

The following diagram therefore commutes up to natural equivalence (by Definition 5.3 we have $U \circ Q = q \circ J$)

$$\begin{tikzcd}
\mathbb{K}(\mathfrak{Qco}X) \arrow[r, "q"] \arrow[d, "J_{\rho}"'] & \mathbb{D}(\mathfrak{Qco}X) \\
\mathbb{K}(\mathrm{Flat}\, X) \arrow[r, "Q"'] & \mathbb{K}_m(\mathrm{Proj}\, X) \arrow[u, "U"']
\end{tikzcd}$$

By Theorem 5.5 the functor U has a left adjoint. It follows that if either of the functors in (A.11) has a left adjoint, then q has a left adjoint. This implies that products are exact in $\mathfrak{Qco}(X)$, completing the proof. ☐

Remark A.15. In particular, this shows that for a field k neither of the functors

$$\mathbb{K}(\mathrm{Flat}\,\mathbb{P}^1_k) \longrightarrow \mathbb{D}(\mathfrak{Qco}\mathbb{P}^1_k), \quad \mathbb{K}(\mathrm{Flat}\,\mathbb{P}^1_k) \longrightarrow \mathbb{K}_m(\mathrm{Proj}\,\mathbb{P}^1_k)$$

admits a left adjoint, because products in $\mathfrak{Qco}(\mathbb{P}^1_k)$ are not exact [Kra05, Example 4.9].

Appendix B

Flat Function Objects

We define the closed monoidal structure on the homotopy category $\mathbb{K}(\operatorname{Flat} X)$ of flat quasi-coherent sheaves, using the adjoint constructed in Appendix A. This result is applied to give an alternative description of the function objects in $\mathbb{K}_m(\operatorname{Proj} X)$ and to define a closed monoidal structure on $\mathbb{K}(\operatorname{Inj} X)$ in the presence of a dualizing complex.

Setup. In this appendix X is a noetherian scheme and sheaves are defined over X.

We make use of two functors, the inclusion and quotient, respectively

$$J : \mathbb{K}(\operatorname{Flat} X) \longrightarrow \mathbb{K}(\mathfrak{Qco} X), \quad Q : \mathbb{K}(\operatorname{Flat} X) \longrightarrow \mathbb{K}_m(\operatorname{Proj} X)$$

Both functors admit right adjoints, by Theorem A.13 and Theorem 3.16, and these adjoints are denoted by J_ρ and Q_ρ as per our usual notational conventions. The tensor product of flat quasi-coherent sheaves is flat, so the tensor product on $\mathbb{K}(\mathfrak{Qco} X)$ restricts to $\mathbb{K}(\operatorname{Flat} X)$, making it into a tensor triangulated category; see Chapter 6. The corresponding closed structure, denoted by $\mathscr{F}\!lat(-,-)$, is defined as follows.

Definition B.1. Let $\mathscr{F}, \mathscr{G}$ be complexes of quasi-coherent sheaves, and $\mathscr{H}om_{\mathrm{qc}}(\mathscr{F}, \mathscr{G})$ the corresponding function object in $\mathbb{K}(\mathfrak{Qco} X)$, given by Proposition 6.17. We define

$$\mathscr{F}\!lat(\mathscr{F}, \mathscr{G}) = J_\rho \mathscr{H}om_{\mathrm{qc}}(\mathscr{F}, \mathscr{G})$$

which is a complex of flat quasi-coherent sheaves.

We refer the reader to [HPS97, Definition A.2.1] for the definition of a closed monoidal structure *compatible with the triangulation.*

Proposition B.2. *The triangulated category* $\mathbb{K}(\operatorname{Flat} X)$ *is closed symmetric monoidal. It has a tensor product and function object* $\mathscr{F}\!lat(-,-)$ *compatible with the triangulation, and there is a natural isomorphism*

$$\operatorname{Hom}_{\mathbb{K}(\operatorname{Flat} X)}(\mathscr{F} \otimes \mathscr{G}, \mathscr{H}) \xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}(\operatorname{Flat} X)}(\mathscr{F}, \mathscr{F}\!lat(\mathscr{G}, \mathscr{H})) \tag{B.1}$$

Proof. From Definition B.1 it is clear that $\mathscr{F}\!lat(-,-)$ is functorial, contravariantly in the first variable and covariantly in the second, and is a triangulated functor in both variables.

We already know that the closed symmetric monoidal structure on $\mathbb{K}(\mathfrak{Qco}X)$ is compatible with the triangulation (Proposition 6.17), and it follows that the same is true of $\mathbb{K}(\operatorname{Flat} X)$. The adjunction isomorphism (B.1) is an easy consequence of the adjunction isomorphism between J and J_ρ and the corresponding isomorphism for $\mathbb{K}(\mathfrak{Qco}X)$. $\square$

In Chapter 6 we defined the function object in $\mathbb{K}_m(\operatorname{Proj} X)$ via Brown representability. Using Appendix A we can give a description more in line with the usual definition of the derived Hom in the derived category of modules over a ring. First, a technical observation. Recall that $\mathbb{E}(X) \subseteq \mathbb{K}(\operatorname{Flat} X)$ is the subcategory of acyclic, $\mathbb{K}$-flat complexes.

Lemma B.3. *Given complexes $\mathscr{F}, \mathscr{C}$ of flat quasi-coherent sheaves with $\mathscr{C}$ in $\mathbb{E}(X)^\perp$, the complex $\mathscr{F}\!lat(\mathscr{F}, \mathscr{C})$ belongs to $\mathbb{E}(X)^\perp$.*

Proof. Given $\mathscr{E}$ in $\mathbb{E}(X)$ we have $\operatorname{Hom}(\mathscr{E}, \mathscr{F}\!lat(\mathscr{F}, \mathscr{C})) \xrightarrow{\sim} \operatorname{Hom}(\mathscr{E} \otimes \mathscr{F}, \mathscr{C})$ which is zero because, by Lemma 6.1, the tensor product $\mathscr{E} \otimes \mathscr{F}$ belongs to $\mathbb{E}(X)$. $\square$

Proposition B.4. *The function object in $\mathbb{K}_m(\operatorname{Proj} X)$ can be defined by*

$$\mathbb{R}\mathscr{F}\!lat(\mathscr{F}, \mathscr{G}) = \mathscr{F}\!lat(\mathscr{F}, Q_\rho(\mathscr{G})) \tag{B.2}$$

Proof. The function object was defined in Proposition 6.2 to be the right adjoint to the tensor product, so the following calculation shows that $\mathbb{R}\mathscr{F}\!lat(-,-)$ can be defined by the construction in (B.2). We have a natural isomorphism

$$\begin{aligned}
\operatorname{Hom}_{\mathbb{K}_m(\operatorname{Proj} X)}(\mathscr{F} \otimes \mathscr{G}, \mathscr{H}) &\xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}(\operatorname{Flat} X)}(\mathscr{F} \otimes \mathscr{G}, Q_\rho(\mathscr{H})) && \text{(Adjunction)}\\
&\xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}(\operatorname{Flat} X)}(\mathscr{F}, \mathscr{F}\!lat(\mathscr{G}, Q_\rho(\mathscr{H}))) && \text{(Adjunction)}\\
&\xrightarrow{\sim} \operatorname{Hom}_{\mathbb{K}_m(\operatorname{Proj} X)}(\mathscr{F}, \mathscr{F}\!lat(\mathscr{G}, Q_\rho(\mathscr{H}))) && \text{(Lemma B.3)}
\end{aligned}$$

which completes the proof. $\square$

Lemma B.5. *There is a canonical morphism in $\mathbb{K}_m(\operatorname{Proj} X)$ natural in both variables*

$$\mathscr{F}\!lat(\mathscr{F}, \mathscr{G}) \longrightarrow \mathbb{R}\mathscr{F}\!lat(\mathscr{F}, \mathscr{G}) \tag{B.3}$$

which is an isomorphism if $\mathscr{G}$ belongs to $\mathbb{E}(X)^\perp$.

Proof. From the adjunction between Q and Q_ρ we have a unit transformation $1 \longrightarrow Q_\rho Q$, which determines a morphism natural in both variables $\mathscr{F}\!lat(-,-) \longrightarrow \mathscr{F}\!lat(-, Q_\rho Q(-))$ as required. If $\mathscr{G}$ belongs to $\mathbb{E}(X)^\perp$ then $\mathscr{G} \longrightarrow Q_\rho Q(\mathscr{G})$ is an isomorphism in $\mathbb{K}(\operatorname{Flat} X)$, so the canonical morphism $\mathscr{F}\!lat(\mathscr{F}, \mathscr{G}) \longrightarrow \mathbb{R}\mathscr{F}\!lat(\mathscr{F}, \mathscr{G})$ is an isomorphism, as claimed. $\square$

Assume that X has a dualizing complex $\mathscr{D}$, which is always assumed to be a bounded complex of injective quasi-coherent sheaves. By Theorem 8.4 there is an equivalence

$$- \otimes \mathscr{D} : \mathbb{K}_m(\operatorname{Proj} X) \xrightarrow{\sim} \mathbb{K}(\operatorname{Inj} X) \tag{B.4}$$

the quasi-inverse of which is described by Proposition 8.9 as the functor

$$\mathscr{H}\!om_{\mathrm{qc}}(\mathscr{D}, -) : \mathbb{K}(\operatorname{Inj} X) \xrightarrow{\sim} \mathbb{K}_m(\operatorname{Proj} X) \tag{B.5}$$

From the closed monoidal structure on $\mathbb{K}_m(\operatorname{Proj} X)$, described in Chapter 6, we obtain an induced structure on $\mathbb{K}(\operatorname{Inj} X)$, with $\mathscr{D}$ the unit object of the tensor product.

Proposition B.6. *If X admits a dualizing complex $\mathscr{D}$ then $\mathbb{K}(\operatorname{Inj} X)$ is closed symmetric monoidal: it has tensor product $- \otimes_{\mathrm{Inj}} -$ and function object $\mathscr{I}nj(-,-)$ defined by*

$$\mathscr{I} \otimes_{\mathrm{Inj}} \mathscr{I}' = \mathscr{I} \otimes \mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, \mathscr{I}')$$
$$\mathscr{I}nj(\mathscr{I}, \mathscr{I}') = \mathscr{D} \otimes \mathscr{F}lat(\mathscr{I}, \mathscr{I}')$$

which are compatible with the triangulation.

Proof. For any complex $\mathscr{J}$ of injective quasi-coherent sheaves we have by Proposition 8.9 a natural isomorphism in $\mathbb{K}(\operatorname{Inj} X)$

$$\mathscr{J} \xrightarrow{\sim} \mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, \mathscr{J}) \otimes \mathscr{D} \tag{B.6}$$

Let $\mathscr{I}$ and $\mathscr{I}'$ be complexes of injective quasi-coherent sheaves. The tensor product on $\mathbb{K}(\operatorname{Inj} X)$ induced by the equivalence of Theorem 8.4 is given by

$$\mathscr{I} \otimes_{\mathrm{Inj}} \mathscr{I}' = \mathscr{D} \otimes \big(\mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, \mathscr{I}) \otimes \mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, \mathscr{I}')\big) \cong \mathscr{I} \otimes \mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, \mathscr{I}') \tag{B.7}$$

Applying (B.6) to $\mathscr{I}'$ instead of $\mathscr{I}$ in the above, we find that the definition of the tensor product is actually symmetric. The function object in $\mathbb{K}(\operatorname{Inj} X)$ is defined by

$$\mathscr{I}nj(\mathscr{I}, \mathscr{I}') = \mathscr{D} \otimes \mathbb{R}\mathscr{F}lat(\mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, \mathscr{I}), \mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, \mathscr{I}'))$$

This needs some simplification. By Lemma 8.8 the complex $\mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, \mathscr{I}')$ belongs to the orthogonal $\mathbb{E}(X)^{\perp}$ so we have an isomorphism

$$\begin{aligned} \mathscr{I}nj(\mathscr{I}, \mathscr{I}') &\xrightarrow{\sim} \mathscr{D} \otimes \mathscr{F}lat(\mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, \mathscr{I}), \mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, \mathscr{I}')) && \text{(Lemma B.5)}\\ &\xrightarrow{\sim} \mathscr{D} \otimes \mathscr{F}lat(\mathscr{H}om_{\mathrm{qc}}(\mathscr{D}, \mathscr{I}) \otimes \mathscr{D}, \mathscr{I}') && \text{(Adjunction)}\\ &\xrightarrow{\sim} \mathscr{D} \otimes \mathscr{F}lat(\mathscr{I}, \mathscr{I}') && \text{(By (B.6))} \end{aligned}$$

We already know that $\mathbb{K}_m(\operatorname{Proj} X)$ is closed symmetric monoidal, and that this structure is compatible with the triangulation, so this completes the proof. Note that it really should be possible to replace $\mathscr{F}lat(\mathscr{I}, \mathscr{I}')$ by $\mathscr{H}om_{\mathrm{qc}}(\mathscr{I}, \mathscr{I}')$ after some technical improvements; see the next remark. □

Remark B.7. Over a noetherian ring A any product of flat modules is flat. For complexes I and I' of injective A-modules, $\operatorname{Hom}_A(I, I')$ is a complex of flat modules and we have an isomorphism $\operatorname{Flat}(I, I') \xrightarrow{\sim} \operatorname{Hom}_A(I, I')$. Thus the function object in $\mathbb{K}(\operatorname{Inj} A)$ has the form $\operatorname{Inj}(I, I') = D \otimes_A \operatorname{Hom}_A(I, I')$. I do not know if the same is true of schemes; that is, given a noetherian scheme X, I do not know if flat quasi-coherent sheaves are closed under products in the category $\mathfrak{Qco}(X)$.

Appendix C

Function Objects from Brown Representability

Let $\mathcal{T}$ be a tensor triangulated category. This is an additive category with the structure of a triangulated category and the structure of a symmetric monoidal category, such that the tensor product $- \otimes -$ respects the triangulation; see [MVW06, Definition 8A.1]. If cohomological functors on $\mathcal{T}$ are representable and the tensor product commutes with coproducts, then we obtain function objects in $\mathcal{T}$ for free: simply define $\mathrm{Map}(x, y)$ to be the object representing the cohomological functor

$$z \mapsto \mathrm{Hom}_{\mathcal{T}}(z \otimes x, y)$$

This makes $\mathcal{T}$ into a closed symmetric monoidal category, but it is not immediately clear that the closed structure respects the triangulation. While it is well-known that $\mathrm{Map}(x, -)$ is a triangulated functor (it is the right adjoint of the triangulated functor $- \otimes x$) the argument which shows that $\mathrm{Map}(-, -)$ is triangulated in the first variable does not appear to be widely known. In this appendix we give the proof, assuming one mild compatibility condition on the tensor product introduced by May in [May01]. It is a pleasure to thank Amnon Neeman for explaining the result to us, and kindly allowing us to include it here.

Before proceeding, it is worth making some general comments about the state of tensor triangulated categories in the literature. The definition we adopt from [MVW06] is what everybody agrees on: the tensor product is triangulated in each variable, and there is some bookkeeping involving signs and the suspension; see also [HPS97, Definition A.2.1]. This is enough for many applications, but there are further properties of the tensor product in the natural examples that one could consider adding as axioms. The first person to really take this seriously was May, who in [May01] gives several additional axioms (TC1)-(TC5). A later article by Keller and Neeman [KN02] sheds further light on May's axioms. Assuming that $\mathcal{T}$ satisfies May's axiom (TC3) we will prove that $\mathrm{Map}(-, -)$ is triangulated in both variables. To be precise, we prove the following for any tensor triangulated category $\mathcal{T}$.

Theorem C.1 (Neeman)**.** *Suppose that $\mathcal{T}$ satisfies* (TC3) *and that every cohomological functor on $\mathcal{T}$ is representable. If the tensor product in $\mathcal{T}$ commutes with coproducts, then $\mathcal{T}$ has a closed structure* $\mathrm{Map}(-, -)$ *compatible with the triangulation.*

When we say that the closed monoidal structure is compatible with the triangulation, we mean it in the sense of [HPS97, Definition A.2.1] and when we say that a functor is *cohomological* we mean that it is a *cohomology functor* in the sense of [HPS97, Definition 1.1.3]. This theorem resolves a difficulty pointed out in [HPS97, Remark 1.4.11].

The example we care about is the mock homotopy category $\mathbb{K}_m(\operatorname{Proj} X)$ of projectives. The monoidal structure on this category is simple, and using the theorem we deduce the closed structure very cheaply. A word on the structure of this appendix: to begin with we keep the arguments general, then in Section C.1 we check that the results apply to our examples of interest, $\mathbb{K}_m(\operatorname{Proj} X)$ and $\mathbb{D}(\mathfrak{Qco} X)$. We will make heavy use of the concept of a homotopy pushout, so for the reader's convenience we include the definition.

Definition C.2. [Nee01b, §1.4] Let $\mathcal{T}$ be a triangulated category. A commutative diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "g"'] & Z \arrow[d, "g'"] \\ Y' \arrow[r, "f'"'] & Z' \end{tikzcd} \tag{C.1}$$

is a *homotopy pushout* if there is a triangle in $\mathcal{T}$ of the following form

$$Y \xrightarrow{\begin{pmatrix} g \\ -f \end{pmatrix}} Y' \oplus Z \xrightarrow{\begin{pmatrix} f' & g' \end{pmatrix}} Z' \longrightarrow \Sigma Y \tag{C.2}$$

What we call a *triangle* in this thesis is sometimes qualified as a *distinguished triangle* in the literature. There is a weaker notion of an *exact triangle*, and if in (C.2) we have only an exact triangle, then we say that (C.1) is a *pushpull square*; see [May01, Definition 3.5]. Any homotopy pushout is a pushpull square. Any pushpull square (C.1) has the property that, given morphisms $a : Y' \longrightarrow Q, b : Z \longrightarrow Q$ with $a \circ g = b \circ f$, there is a (non-unique) morphism $\theta : Z' \longrightarrow Q$ with $a = \theta \circ f'$ and $b = \theta \circ g'$.

To explain the axiom (TC3) of May that we need, consider the following situation: let $\mathcal{T}$ be a tensor triangulated category, and suppose we are given two triangles

$$x \longrightarrow y \longrightarrow z \longrightarrow \Sigma x \tag{C.3}$$

$$x' \longrightarrow y' \longrightarrow z' \longrightarrow \Sigma x' \tag{C.4}$$

We can form the *tensor product* of these triangles. This is a diagram in which all columns and rows are triangles, and every square commutes except for $(\star)$ which anticommutes

$$\begin{tikzcd} x \otimes x' \arrow[r] \arrow[d] & y \otimes x' \arrow[r] \arrow[d] & z \otimes x' \arrow[r] \arrow[d] & \Sigma x \otimes x' \arrow[d] \\ x \otimes y' \arrow[r] \arrow[d] & y \otimes y' \arrow[r] \arrow[d] & z \otimes y' \arrow[r] \arrow[d] & \Sigma x \otimes y' \arrow[d] \\ x \otimes z' \arrow[r] \arrow[d] & y \otimes z' \arrow[r] \arrow[d] & z \otimes z' \arrow[r] \arrow[d] \arrow[dr, phantom, "(\star)"] & \Sigma x \otimes z' \arrow[d, "-"] \\ \Sigma x \otimes x' \arrow[r] & \Sigma y \otimes x' \arrow[r] & \Sigma z \otimes x' \arrow[r, "-"'] & \Sigma^2 x \otimes x' \end{tikzcd} \tag{C.5}$$

In the natural examples it is possible to associate an additional object w and several morphisms with this diagram, satisfying some conditions. May axiomatizes this situation, in two equivalent ways, in his axioms (TC3) and (TC3'). It is actually his presentation (TC3') of the axiom that is closest to our needs, so it is what we will state.

Definition C.3. Following May [May01, §4] we say that a tensor triangulated category $\mathcal{T}$ *satisfies* (TC3) if, for any pair of triangles (C.3), (C.4) as above, their tensor product (C.5) comes with *Verdier structure*, by which we mean that there exists an object w and six morphisms $k_1, k_2, k_3, q_1, q_2, q_3$ fitting into a commutative diagram

$$\begin{tikzcd}
 & & \Sigma^{-1}w \arrow[r, "\Sigma^{-1}q_3"] \arrow[d, "\Sigma^{-1}q_2"'] \arrow[dr, phantom, "(1)"] & \Sigma^{-1}y\otimes z' \arrow[d] \arrow[dr] & \\
 & \Sigma^{-1}w \arrow[r, "\Sigma^{-1}q_2"] \arrow[d, "\Sigma^{-1}q_1"'] \arrow[dr, phantom, "(2)"] & x\otimes x' \arrow[r] \arrow[d] \arrow[dr] & y\otimes x' \arrow[r] \arrow[d] \arrow[dr, phantom, "(4)"] & z\otimes x' \arrow[d, "k_3"] \\
\Sigma^{-1}w \arrow[r, "\Sigma^{-1}q_1"] \arrow[d, "\Sigma^{-1}q_3"'] \arrow[dr, phantom, "(3)"] & \Sigma^{-1}z\otimes y' \arrow[r] \arrow[d] \arrow[dr] & x\otimes y' \arrow[r] \arrow[d] \arrow[dr, phantom, "(5)"] & y\otimes y' \arrow[r, "k_2"'] \arrow[d, "k_2"] & w \\
\Sigma^{-1}y\otimes z' \arrow[r] & \Sigma^{-1}z\otimes z' \arrow[r] \arrow[d, "-"'] \arrow[dr, phantom, "(6)"] & x\otimes z' \arrow[r, "k_1"'] \arrow[d, "k_1"] & w & \\
 & z\otimes x' \arrow[r, "k_3"'] & w & &
\end{tikzcd}$$

We also require that various other diagrams commute, and that certain triangles exist, but we direct the reader to the statement of [May01, Lemma 4.7] for the list. The diagram above is all that we need to prove Theorem C.1. If further we can arrange for the squares labeled (1)-(6) to be homotopy pushout squares, we say that $\mathcal{T}$ satisfies (TC3+). Note the sign carried by the lefthand side of (6).

Remark C.4. The examples of interest to us will satisfy (TC3+), but we will only need the weaker (TC3) to prove the main theorem. The emphasis on homotopy pushouts can be traced to the approach of Keller and Neeman in [KN02] where the large diagram in Definition C.3 is given as ($\star\star$) on p.547.

Let $\mathcal{T}$ be a tensor triangulated category with the property that cohomological functors on $\mathcal{T}$ are representable, and the tensor product commutes with coproducts. Given objects x and y in $\mathcal{T}$ we define $\mathrm{Map}(x, y)$ to be the object of $\mathcal{T}$ that represents the cohomological functor $z \mapsto \mathrm{Hom}_{\mathcal{T}}(z \otimes x, y)$. In the standard way we make $\mathrm{Map}(-, -)$ into a bifunctor contravariant in the first variable, such that the following isomorphism is natural

$$\mathrm{Hom}_{\mathcal{T}}(z \otimes x, y) \cong \mathrm{Hom}_{\mathcal{T}}(z, \mathrm{Map}(x, y))$$

This definition makes $\mathcal{T}$ into a closed monoidal category.

Proof of Theorem C.1. For a fixed object $x \in \mathcal{T}$, the functor $\mathrm{Map}(x, -)$ is triangulated, because it is the right adjoint of a triangulated functor. Replacing x by its desuspension

$\Sigma^{-1}x$ we have a natural isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathcal{T}}(z,\mathrm{Map}(\Sigma^{-1}x,y)) &\cong \mathrm{Hom}_{\mathcal{T}}(z\otimes\Sigma^{-1}x,y)\\ &\cong \mathrm{Hom}_{\mathcal{T}}(\Sigma^{-1}(z\otimes x),y)\\ &\cong \mathrm{Hom}_{\mathcal{T}}(z,\Sigma\,\mathrm{Map}(x,y))\end{aligned} \tag{C.6}$$

which yields a natural isomorphism $\mathrm{Map}(\Sigma^{-1}x,y)\cong\Sigma\,\mathrm{Map}(x,y)$. Let τ be the additive inverse of this isomorphism ($-\tau$ makes the diagram coming from (C.6) commute, but the correct morphism makes it *anticommute*, so we change the sign). To complete the proof of the theorem it suffices to check that the pair $(\mathrm{Map}(-,y),\tau)$ defines a triangulated functor $\mathcal{T}^{\mathrm{op}}\longrightarrow\mathcal{T}$, because the other conditions of [HPS97, Definition A.2.1] are already verified. Let a triangle in $\mathcal{T}$ be given

$$x\longrightarrow x'\longrightarrow x''\longrightarrow\Sigma x \tag{C.7}$$

We have a candidate triangle

$$\mathrm{Map}(x'',y)\longrightarrow\mathrm{Map}(x',y)\longrightarrow\mathrm{Map}(x,y)\longrightarrow\Sigma\,\mathrm{Map}(x'',y) \tag{C.8}$$

that we have to show is actually a triangle. The trick is to take the mapping cone of the first morphism in (C.8), and then argue that it agrees with $\mathrm{Map}(x,y)$. That is, we extend to an actual triangle in $\mathcal{T}$

$$\mathrm{Map}(x'',y)\longrightarrow\mathrm{Map}(x',y)\longrightarrow T\longrightarrow\Sigma\,\mathrm{Map}(x'',y)$$

Take the tensor product of this triangle with the original triangle (C.7) to obtain a diagram of the form (C.5). The large diagram is not so important; what we need are the following commutative squares, provided by (TC3)

$$\begin{tikzcd}
\mathrm{Map}(x',y)\otimes x \arrow[r] \arrow[d] \arrow[dr, phantom, "(4)"] & T\otimes x \arrow[d] & \mathrm{Map}(x'',y)\otimes x' \arrow[r] \arrow[d] \arrow[dr, phantom, "(5)"] & \mathrm{Map}(x',y)\otimes x' \arrow[d]\\
\mathrm{Map}(x',y)\otimes x' \arrow[r] & w & \mathrm{Map}(x'',y)\otimes x'' \arrow[r] & w
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma^{-1}T\otimes x'' \arrow[r] \arrow[d, "-"'] \arrow[dr, phantom, "(6)"] & \mathrm{Map}(x'',y)\otimes x'' \arrow[d]\\
T\otimes x \arrow[r] & w
\end{tikzcd}$$

From the adjunction between the tensor product and function object in $\mathcal{T}$, we obtain a counit morphism $\varepsilon:\mathrm{Map}(t,y)\otimes t\longrightarrow y$ for any object $t\in\mathcal{T}$. Moreover, this morphism is natural in t, and in particular the following diagram commutes

$$\begin{tikzcd}
\mathrm{Map}(x'',y)\otimes x' \arrow[r] \arrow[d] & \mathrm{Map}(x',y)\otimes x' \arrow[d, "\varepsilon"]\\
\mathrm{Map}(x'',y)\otimes x'' \arrow[r, "\varepsilon"'] & y
\end{tikzcd} \tag{C.9}$$

By [May01, Lemma 4.9] the commutative squares (4)-(6) provided by (TC3) are pushpull squares, so from (C.9) we deduce $f : w \longrightarrow y$ making the following diagram commute

$$\begin{tikzcd}
\mathrm{Map}(x'', y) \otimes x' \arrow[r] \arrow[d] \arrow[dr, phantom, "(5)"] & \mathrm{Map}(x', y) \otimes x \arrow[d] \arrow[ddr, bend left, "\varepsilon"] & \\
\mathrm{Map}(x'', y) \otimes x'' \arrow[r] \arrow[drr, bend right, "\varepsilon"'] & w \arrow[dr, "f"] & \\
& & y
\end{tikzcd}$$

By adjunction the composite $T \otimes x \longrightarrow w \longrightarrow y$ determines a morphism $T \longrightarrow \mathrm{Map}(x, y)$ fitting into the following diagram, in which the top row is a triangle and the bottom row is a candidate triangle

$$\begin{tikzcd}
\mathrm{Map}(x'', y) \arrow[r] \arrow[d, "1"'] & \mathrm{Map}(x', y) \arrow[r] \arrow[d, "1"'] \arrow[dr, phantom, "(i)"] & T \arrow[r] \arrow[d] \arrow[dr, phantom, "(ii)"] & \Sigma\,\mathrm{Map}(x'', y) \arrow[d, "1"] \\
\mathrm{Map}(x'', y) \arrow[r] & \mathrm{Map}(x', y) \arrow[r] & \mathrm{Map}(x, y) \arrow[r] & \Sigma\,\mathrm{Map}(x'', y)
\end{tikzcd} \tag{C.10}$$

We deduce commutativity of (i) from commutativity of (4), but commutativity of (ii) is more subtle. To prove it, it suffices to establish that applying $\mathrm{Hom}_{\mathcal{T}}(t, -)$ to (ii) leaves a commutative diagram of abelian groups for $t \in \mathcal{T}$. By the adjunction, this diagram of abelian groups is

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{T}}(t, T) \arrow[d] \arrow[dr, bend left] & \\
\mathrm{Hom}_{\mathcal{T}}(t \otimes x, y) \arrow[r] & \mathrm{Hom}_{\mathcal{T}}(t \otimes \Sigma^{-1} x'', y)
\end{tikzcd}$$

Chasing a morphism $t \longrightarrow T$ around this diagram, we conclude that commutativity of (ii) follows from commutativity of (6), so (C.10) is a morphism of candidate triangles. Applying $\mathrm{Hom}_{\mathcal{T}}(t, -)$ to this morphism of candidate triangles yields a morphism of complexes of abelian groups, the top row of which is exact because the top row of (C.10) is a triangle. The bottom row is exact because of the adjunction isomorphism

$$\mathrm{Hom}_{\mathcal{T}}(t, \mathrm{Map}(-, y)) \cong \mathrm{Hom}_{\mathcal{T}}(t \otimes -, y)$$

From the Five Lemma we deduce that $T \longrightarrow \mathrm{Map}(x, y)$ is an isomorphism, so the bottom row of (C.10) is a triangle and the proof is complete. $\square$

C.1 Examples satisfying May's axiom

We prove that, given a symmetric monoidal abelian category $\mathcal{A}$, the homotopy category $\mathbb{K}(\mathcal{A})$ is a tensor triangulated category satisfying the axiom (TC3+) of the previous section (see Definition C.3). It will follow easily that the mock homotopy category $\mathbb{K}_m(\mathrm{Proj}\, X)$ and the derived category $\mathbb{D}(\mathfrak{Qco}\, X)$ satisfy this axiom for any scheme. The reader can find similar arguments in [May01, §6] and [KN02, §3], but one of our results (Proposition 6.2) depends crucially on these facts, so we feel it is worth including a detailed proof here.

Setup. In this section $\mathcal{A}$ is an abelian category with a tensor product $-\otimes-$ making it into a symmetric monoidal category. We assume that the tensor product commutes with coproducts. The examples we have in mind are categories of modules over a commutative ring, and quasi-coherent sheaves over a scheme.

The structure that makes $\mathbb{K}(\mathcal{A})$ into a triangulated category is the mapping cone. In verifying that $\mathbb{K}(\mathcal{A})$ satisfies (TC3+) we will be inevitably drawn into some technical details involving mapping cones and homotopies; since sign errors are the bane of such verifications, we remind the reader of our conventions: given a morphism $u : x \longrightarrow y$ of complexes, the *mapping cone* is the complex $\mathrm{cone}(u)$ with differential

$$\partial^n = \begin{pmatrix} -\partial_x^{n+1} & 0 \\ u^{n+1} & \partial_y^n \end{pmatrix} : x^{n+1} \oplus y^n \longrightarrow x^{n+2} \oplus y^{n+1}$$

Next we recall some definitions.

Definition C.5. A morphism of triangles in a triangulated category $\mathcal{T}$ is a commutative diagram in $\mathcal{T}$, with triangles for rows

$$\begin{tikzcd} x \arrow[r] \arrow[d, "f"'] & y \arrow[r] \arrow[d, "g"'] & y \arrow[r] \arrow[d, "h"'] & \Sigma x \arrow[d, "\Sigma f"] \\ x' \arrow[r] & y' \arrow[r] & z' \arrow[r] & \Sigma x' \end{tikzcd}$$

We say that this is a *good* morphism of triangles if its mapping cone is a triangle; this idea is due to Neeman, and is explained at length in [Nee01b, §1.3].

Example C.6. Suppose that we have a commutative diagram of complexes in $\mathcal{A}$

$$\begin{tikzcd} x \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "D"] & y \arrow[d, "g"] \\ x' \arrow[r, "u'"'] & y' \end{tikzcd}$$

The morphism of complexes $\mathrm{cone}(D) : \mathrm{cone}(u) \longrightarrow \mathrm{cone}(u')$ given by $\mathrm{cone}(D)^n = f^{n+1} \oplus g^n$ makes the following diagram a good morphism of triangles

$$\begin{tikzcd} x \arrow[r, "u"] \arrow[d, "f"'] & y \arrow[r] \arrow[d, "g"'] & \mathrm{cone}(u) \arrow[r] \arrow[d, "\mathrm{cone}(D)"'] & \Sigma x \arrow[d, "\Sigma f"] \\ x' \arrow[r, "u'"'] & y' \arrow[r] & \mathrm{cone}(u') \arrow[r] & \Sigma x' \end{tikzcd}$$

Example C.7. Suppose that we have a commutative diagram of complexes in $\mathcal{A}$

$$\begin{tikzcd} 0 \arrow[r] & x \arrow[r, "a"] \arrow[d, "f"'] & y \arrow[r, "b"] \arrow[d, "g"'] & z \arrow[r] \arrow[d, "h"'] & 0 \\ 0 \arrow[r] & x' \arrow[r, "a'"'] & y' \arrow[r, "b'"'] & z' \arrow[r] & 0 \end{tikzcd}$$

where the rows are degree-wise split exact. By Lemma 2.15 there are canonical morphisms $z : z \longrightarrow \Sigma x$ and $z' : z' \longrightarrow \Sigma x'$ in $\mathbb{K}(\mathcal{A})$, making a morphism of triangles in $\mathbb{K}(\mathcal{A})$

$$\begin{tikzcd} x \arrow[r, "a"] \arrow[d, "f"'] & y \arrow[r, "b"] \arrow[d, "g"'] & z \arrow[r, "-z"] \arrow[d, "h"'] & \Sigma x \arrow[d, "\Sigma f"] \\ x' \arrow[r, "a'"'] & y' \arrow[r, "b'"'] & z' \arrow[r, "-z'"'] & \Sigma x' \end{tikzcd} \tag{C.11}$$

Comparing this morphism of triangles with the one in Example C.6, we find after a short calculation that (C.11) is also a good morphism of triangles.

Lemma C.8. *Let $\mathcal{T}$ be a triangulated category, and suppose that we have a good morphism of triangles*

$$\begin{tikzcd} x \arrow[r] \arrow[d, "f"'] & y \arrow[r] \arrow[d, "g"'] \arrow[dr, phantom, "\mathrm{(I)}"] & z \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{(II)}"] & \Sigma x \arrow[d] \\ x' \arrow[r] & y' \arrow[r] & z' \arrow[r] & \Sigma x' \end{tikzcd} \tag{C.12}$$

If $f = 1$ then (I) *is a homotopy pushout, and if $g = 1$ then* (II) *is a homotopy pushout.*

Proof. To prove that a square is a homotopy pushout, we have to prove that a triangle of a certain type exists. In the situation where $f = 1$ (resp. $g = 1$) a suitable triangle exists as a direct summand of the mapping cone of (C.12), which is a triangle by assumption. Any direct summand of a triangle is a triangle; see the proof of [Nee01b, Lemma 1.4.3]. □

To prove that (TC3+) holds for $\mathbb{K}(\mathcal{A})$, we need to construct six homotopy pushouts. The idea is to construct good morphisms of triangles using Example C.7, and apply Lemma C.8 to these morphisms to produce the desired homotopy pushouts.

Proposition C.9. *Suppose that we have a commutative diagram of complexes in $\mathcal{A}$*

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "gf"'] & y \arrow[d, "g"] \\ & y' \end{tikzcd}$$

where g and f are degree-wise split monomorphisms. There is an induced degree-wise split exact sequence of complexes

$$0 \longrightarrow y/x \longrightarrow y'/x \longrightarrow y'/y \longrightarrow 0 \tag{C.13}$$

If we extend f, g, gf and (C.13) *canonically to triangles in $\mathbb{K}(\mathcal{A})$, using Lemma 2.15*

$$x \xrightarrow{f} y \longrightarrow y/x \xrightarrow{-z} \Sigma x$$

$$x \xrightarrow{gf} y' \longrightarrow y'/x \xrightarrow{-z'} \Sigma x$$

$$y \xrightarrow{g} y' \longrightarrow y'/y \xrightarrow{-z''} \Sigma y$$

$$y/x \longrightarrow y'/x \longrightarrow y'/y \xrightarrow{-z'''} \Sigma y/x$$

then these triangles fit into a commutative diagram in $\mathbb{K}(\mathcal{A})$

$$\begin{tikzcd}
 & & & & \Sigma x \arrow[ddr, "1"] & & \\
 & & & y/x \arrow[ur] \arrow[ddr] \arrow[dd, phantom, "(\alpha)"] & & & \\
 & & & & & \Sigma x \arrow[dddddr, "\Sigma f"] & \\
 & & & & y'/x \arrow[ur] \arrow[dddr] \arrow[ddrr, phantom, "(\beta)"] & & \\
 & y \arrow[uuurr] \arrow[dr, "g"] & & & & & \\
 & & y' \arrow[uurr] \arrow[drrr] & & & & \\
x \arrow[uur, "f"] \arrow[urr, "gf"'] & & & & & y'/y \arrow[dr] \arrow[dd] & \\
 & & & & & & \Sigma y \arrow[dl] \\
 & & & & & \Sigma y/x &
\end{tikzcd}$$

in which (α) *and* (β) *are homotopy pushouts.*

Proof. We have commutative diagrams of complexes with degree-wise split exact rows

$$\begin{tikzcd}
0 \arrow[r] & x \arrow[r, "f"] \arrow[d, "1"] & y \arrow[r] \arrow[d, "g"] & y/x \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & x \arrow[r, "gf"'] & y' \arrow[r] & y'/x \arrow[r] & 0
\end{tikzcd} \tag{C.14}$$

$$\begin{tikzcd}
0 \arrow[r] & x \arrow[r, "gf"] \arrow[d, "f"] & y' \arrow[r] \arrow[d, "1"] & y'/x \arrow[r] \arrow[d, "i"] & 0 \\
0 \arrow[r] & y \arrow[r, "g"'] & y' \arrow[r] & y'/y \arrow[r] & 0
\end{tikzcd} \tag{C.15}$$

We deduce that h is a split monomorphism in each degree, so there is a third morphism of degree-wise split exact sequences of complexes

$$\begin{tikzcd}
0 \arrow[r] & y \arrow[r, "g"] \arrow[d] & y' \arrow[r] \arrow[d] & y'/y \arrow[r] \arrow[d, "1"] & 0 \\
0 \arrow[r] & y/x \arrow[r, "h"'] & y'/x \arrow[r, "i"'] & y'/y \arrow[r] & 0
\end{tikzcd} \tag{C.16}$$

Applying Example C.7 to the diagrams (C.14-C.16) produces the desired large diagram, and Lemma C.8 yields that (α) and (β) are homotopy pushouts. $\square$

We will need the following variant, where g is a degree-wise split *epimorphism.*

Proposition C.10. *Suppose that we have a commutative diagram of complexes in* $\mathcal{A}$

$$\begin{tikzcd}
x \arrow[r, "f"] \arrow[dr, "gf"'] & y \arrow[d, "g"] \\
 & y'
\end{tikzcd}$$

where f, gf are degree-wise split monomorphisms, and g a degree-wise split epimorphism. There are degree-wise split exact sequences of complexes, where $k = Ker(g)$

$$0 \longrightarrow k \xrightarrow{t} y \xrightarrow{g} y' \longrightarrow 0 \tag{C.17}$$

$$0 \longrightarrow k \xrightarrow{t'} y/x \xrightarrow{h} y'/x \longrightarrow 0 \tag{C.18}$$

If we extend f, gf and (C.17, C.18) canonically to triangles in $\mathbb{K}(\mathcal{A})$, using Lemma 2.15

$$x \xrightarrow{f} y \longrightarrow y/x \xrightarrow{-z} \Sigma x$$

$$x \xrightarrow{gf} y' \longrightarrow y'/x \xrightarrow{-z'} \Sigma x$$

$$k \xrightarrow{t} y \xrightarrow{g} y' \xrightarrow{-w} \Sigma x$$

$$k \xrightarrow{t'} y/x \xrightarrow{h} y'/x \xrightarrow{-w'} \Sigma k$$

then these triangles fit into a commutative diagram in $\mathbb{K}(\mathcal{A})$

$$\begin{tikzcd}
 & & & & \Sigma x \arrow[ddr, "1"] & & \\
 & & & y/x \arrow[ur] \arrow[ddr, "h"] \arrow[dd, phantom, "\alpha"] & & & \\
 & & & & & \Sigma x \arrow[dddddr, "\Sigma f"] \arrow[dd, phantom, "\beta"] & \\
 & & & \; & y'/x \arrow[ur] \arrow[dddr, "w'"] & & \\
 & y \arrow[uuurr] \arrow[dr, "g"] & & & & \; & \\
 & & y' \arrow[uurr] \arrow[drrr, "w"] & & & & \\
 & & & & & \Sigma k \arrow[dr, "\Sigma t"] \arrow[dd, "\Sigma t'"] & \\
x \arrow[uuur, "f"] \arrow[uurr, "gf"] & & & & & & \Sigma y \arrow[dl] \\
 & & & & & \Sigma y/x &
\end{tikzcd}$$

in which (α) and (β) are homotopy pushouts.

Proof. We have a commutative diagram of complexes, where the rows are degree-wise split exact sequences

$$\begin{tikzcd}
0 \arrow[r] & x \arrow[r, "f"] \arrow[d, "1"] & y \arrow[r] \arrow[d, "g"] & y/x \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & x \arrow[r, "gf"'] & y' \arrow[r] & y'/x \arrow[r] & 0
\end{tikzcd} \tag{C.19}$$

One deduces that h is a degree-wise split epimorphism, and using the Nine Lemma we obtain a commutative diagram of complexes where the rows are degree-wise split exact

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r, "t"] \arrow[d, "1"] & y \arrow[r, "g"] \arrow[d] & y' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & k \arrow[r, "t'"'] & y/x \arrow[r, "h"'] & y'/x \arrow[r] & 0
\end{tikzcd} \tag{C.20}$$

Applying Example C.7 to these two diagrams takes care of commutativity in the larger diagram of every square except for the one marked (β). Commutativity of this square will be established at the same time that we check it is a homotopy pushout: we claim that the following diagram is a good morphism of triangles in $\mathbb{K}(\mathcal{A})$

$$\begin{tikzcd} x \arrow[r, "gf"] \arrow[d, "f"'] \arrow[dr, phantom, "(\mathrm{I})"] & y' \arrow[r] \arrow[d, "1"'] & y'/x \arrow[r, "-z'"] \arrow[d, "w'"'] \arrow[dr, phantom, "(\beta)"] & \Sigma x \arrow[d, "\Sigma f"] \\ y \arrow[r, "g"'] & y' \arrow[r, "w"'] & \Sigma k \arrow[r, "\Sigma t"'] & \Sigma y \end{tikzcd} \tag{C.21}$$

To prove this, compare (C.21) with the good morphism of triangles arising, as in Example C.6, from the commutative square (I). Here z', w, w' are the "connecting morphisms" from degree-wise split exact sequences of complexes. These morphisms are described abstractly in Lemma 2.15, but one can describe them more explicitly by choosing, in each degree, a splitting of the relevant exact sequence. For example: choose a splitting of (C.17) in each degree, given by $s(t)^n : y^n \longrightarrow k^n$ and $s(g)^n : (y')^n \longrightarrow y^n$ such that

$$s(t)^n \circ t^n = 1, \quad g^n \circ s(g)^n = 1, \quad t^n \circ s(t)^n + s(g)^n \circ g^n = 1$$

Then we can choose w (which is, after all, only claimed to be canonical up to homotopy) to be the morphism with $w^n = s(t)^{n+1} \circ \partial_y^n \circ s(g)^n$. If we also choose a splitting $s(f)^n$ of f^n in each degree, such that $s(t)^n \circ f^n$ and $s(f)^n \circ t^n$ both vanish (this is possible: first take splittings of $k \longrightarrow y/x$ and $x \longrightarrow y'$) then it is straightforward to check that there are homotopy equivalences $\mathrm{cone}(gf) \xrightarrow{\sim} y'/x$ and $\mathrm{cone}(g) \xrightarrow{\sim} \Sigma k$ making (C.21) isomorphic to the good morphism of triangles coming from Example C.6. We conclude that (C.21) is good, so by Lemma C.8 the square marked (β) is a homotopy pushout, as claimed. $\square$

It is well-known that, up to isomorphism, every triangle in $\mathbb{K}(\mathcal{A})$ is obtained via Lemma 2.15 from an exact sequence of complexes that is degree-wise split exact (the proof usually goes by way of the mapping cylinder). Therefore, in verifying (TC3+) for $\mathbb{K}(\mathcal{A})$, it suffices to understand the tensor product of two such triangles. For the rest of this section, let us fix two degree-wise split exact sequences of complexes in $\mathcal{A}$

$$0 \longrightarrow x \xrightarrow{f} y \xrightarrow{u} z \longrightarrow 0 \tag{C.22}$$
$$0 \longrightarrow x' \xrightarrow{g} y' \xrightarrow{v} z' \longrightarrow 0 \tag{C.23}$$

We can take the tensor product of these exact sequences, to obtain a commutative diagram in which each row and column is degree-wise split exact

$$\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & x \otimes x' \arrow[r] \arrow[d] & y \otimes x' \arrow[r] \arrow[d] & z \otimes x' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & x \otimes y' \arrow[r] \arrow[d] & y \otimes y' \arrow[r] \arrow[d] & z \otimes y' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & x \otimes z' \arrow[r] \arrow[d] & y \otimes z' \arrow[r] \arrow[d] & z \otimes z' \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd} \tag{C.24}$$

From (C.22) and (C.23) we obtain canonical triangles in $\mathbb{K}(\mathcal{A})$

$$x \xrightarrow{f} y \xrightarrow{u} z \xrightarrow{-w} \Sigma x \tag{C.25}$$

$$x' \xrightarrow{g} y' \xrightarrow{v} z' \xrightarrow{-q} \Sigma x' \tag{C.26}$$

whose tensor product is the diagram (C.5) given at the beginning of this appendix. To establish (TC3+) we have to produce an object w and six morphisms $k_1, k_2, k_3, q_1, q_2, q_3$ that fit into six homotopy pushout squares, marked (1)-(6) in Definition C.3. For our choice of triangles (C.25, C.26) we define the object w to be the cokernel of $f \otimes g : x \otimes x' \longrightarrow y \otimes y'$. Note that we have a degree-wise split exact sequence

$$0 \longrightarrow x \otimes x' \xrightarrow{f \otimes g} y \otimes y' \longrightarrow w \longrightarrow 0$$

Applying Proposition C.9 to the degree-wise split monomorphisms $x \otimes x' \longrightarrow y \otimes x'$ and $y \otimes x' \longrightarrow y \otimes y'$ in the first instance, and $x \otimes x' \longrightarrow x \otimes y'$ and $x \otimes y' \longrightarrow y \otimes y'$ in the second instance, we deduce the following four homotopy pushout squares in $\mathbb{K}(\mathcal{A})$

$$\begin{tikzcd}
y \otimes x' \arrow[r] \arrow[d] \arrow[dr, phantom, "(4)"] & z \otimes x' \arrow[d, "k_3"] \\
y \otimes y' \arrow[r, "k_2"'] & w
\end{tikzcd}
\qquad
\begin{tikzcd}
x \otimes y' \arrow[r] \arrow[d] \arrow[dr, phantom, "(5)"] & y \otimes y' \arrow[d, "k_2"] \\
x \otimes z' \arrow[r, "k_1"'] & w
\end{tikzcd}$$

$$\begin{tikzcd}
\Sigma^{-1} w \arrow[r, "\Sigma^{-1} q_3"] \arrow[d, "\Sigma^{-1} q_2"'] \arrow[dr, phantom, "(1)"] & \Sigma^{-1} y \otimes z' \arrow[d] \\
x \otimes x' \arrow[r] & y \otimes x'
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma^{-1} w \arrow[r, "\Sigma^{-1} q_2"] \arrow[d, "\Sigma^{-1} q_1"'] \arrow[dr, phantom, "(2)"] & x \otimes x' \arrow[d] \\
\Sigma^{-1} z \otimes y' \arrow[r] & x \otimes y'
\end{tikzcd}$$

For the two remaining homotopy pushout squares (3) and (6), observe that we have commutative diagrams of complexes, in which the rows are degree-wise split exact

$$\begin{tikzcd}
0 \arrow[r] & x \otimes x' \arrow[r] \arrow[d, "1"'] & x \otimes y' \arrow[r] \arrow[d] & x \otimes z' \arrow[r] \arrow[d, "h'"'] & 0 \\
0 \arrow[r] & x \otimes x' \arrow[r] & y \otimes y' \arrow[r] & w \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & x \otimes x' \arrow[r] \arrow[d, "1"'] & y \otimes x' \arrow[r] \arrow[d] & z \otimes x' \arrow[r] \arrow[d, "i"'] & 0 \\
0 \arrow[r] & x \otimes x' \arrow[r] & y \otimes y' \arrow[r] & w \arrow[r] & 0
\end{tikzcd}$$

We infer that the morphism marked h' is a degree-wise split monomorphism, and the morphism marked i a degree-wise split epimorphism. The composite ih' is just the morphism $x \otimes z' \longrightarrow y \otimes z'$ from (C.24), which is a degree-wise split monomorphism, so we are in a position to apply Proposition C.10 to construct two homotopy pushout squares

$$\begin{tikzcd}
\Sigma^{-1} w \arrow[r, "\Sigma^{-1} q_1"] \arrow[d, "\Sigma^{-1} q_3"'] \arrow[dr, phantom, "(3)"] & \Sigma^{-1} z \otimes y' \arrow[d] \\
\Sigma^{-1} y \otimes z' \arrow[r] & \Sigma^{-1} z \otimes z'
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma^{-1} z \otimes z' \arrow[r] \arrow[d, "-"'] \arrow[dr, phantom, "(6)"] & x \otimes z' \arrow[d, "k_1"] \\
z \otimes x' \arrow[r, "k_3"'] & w
\end{tikzcd}$$

Note the sign in the second square: we desuspend the morphism $z \otimes z' \longrightarrow \Sigma(z \otimes x')$ and change the sign. This happens in our case because w' occurs without a sign in the large diagram of Proposition C.10.

Proposition C.11. *The tensor triangulated category* $\mathbb{K}(\mathcal{A})$ *satisfies* (TC3+)*.*

Proof. Every triangle in $\mathbb{K}(\mathcal{A})$ is isomorphic to a triangle arising from a degree-wise split exact sequence of complexes, so in verifying (TC3+) we can work exclusively with such triangles. In this case, the discussion above constructs homotopy pushouts (1)-(6) of the necessary form, and when one checks the details of the diagrams produced by Proposition C.9 and Proposition C.10 the other conditions for (TC3') given in [May01, Lemma 4.7] are satisfied. $\square$

Given a tensor triangulated category $\mathcal{T}$, we say that $\mathcal{S}$ is a *tensor triangulated subcategory* of $\mathcal{T}$ if it is a triangulated subcategory closed under the tensor product, that contains the unit object of the tensor product in $\mathcal{T}$. Clearly $\mathcal{S}$ is then a tensor triangulated category.

Lemma C.12. *Let* $\mathcal{T}$ *be a tensor triangulated category satisfying* (TC3+)*, and let* $\mathcal{S}$ *be a tensor triangulated subcategory of* $\mathcal{T}$*. Then* $\mathcal{S}$ *also satisfies* (TC3+)*.*

Proof. Let two triangles in $\mathcal{S}$ be given, and let the object w be part of the Verdier structure on the tensor product of these triangles in $\mathcal{T}$. It is clear from the triangles listed in [May01, Lemma 4.7] that w is an object of $\mathcal{S}$, and it follows that (TC3+) holds for $\mathcal{S}$. $\square$

Proposition C.13. *Given a scheme* X *the tensor triangulated categories*

$$\mathbb{K}(\mathfrak{Qco} X), \quad \mathbb{K}(\operatorname{Flat} X), \quad \mathbb{K}_m(\operatorname{Proj} X), \quad \mathbb{D}(\mathfrak{Qco} X)$$

all satisfy (TC3+)*.*

Proof. We know from Proposition C.11 that $\mathbb{K}(\mathfrak{Qco} X)$ satisfies the axiom, and by Lemma C.12 it must also hold for $\mathbb{K}(\operatorname{Flat} X)$. The quotient $\mathbb{K}_m(\operatorname{Proj} X)$ has the tensor product that descends from $\mathbb{K}(\operatorname{Flat} X)$, and every triangle in the quotient is, up to isomorphism, the image of a triangle in $\mathbb{K}(\operatorname{Flat} X)$, so it follows that $\mathbb{K}_m(\operatorname{Proj} X)$ satisfies (TC3+).

Now consider the subcategory $\mathcal{S} = {}^{\perp}\mathbb{K}_{m,\mathrm{ac}}(\operatorname{Proj} X)$ in $\mathbb{K}_m(\operatorname{Proj} X)$. By Proposition 5.2 this is the full subcategory of $\mathbb{K}$-flat complexes in $\mathbb{K}_m(\operatorname{Proj} X)$, and from Theorem 5.5 we have an equivalence of triangulated categories

$$\mathcal{S} \xrightarrow{\mathrm{inc}} \mathbb{K}_m(\operatorname{Proj} X) \xrightarrow{\mathrm{can}} \mathbb{D}(\mathfrak{Qco} X)$$

As observed in Proposition 6.4, this is an equivalence of tensor triangulated categories. Because $\mathcal{S}$ is a tensor triangulated subcategory of $\mathbb{K}_m(\operatorname{Proj} X)$ it satisfies (TC3+) by Lemma C.12, so the same must be true of $\mathbb{D}(\mathfrak{Qco} X)$. $\square$

Remark C.14. If X is a noetherian scheme then $\mathbb{K}_m(\operatorname{Proj} X)$ is compactly generated (Theorem 4.10) and thus cohomological functors on $\mathcal{T}$ are representable. Tensor products in $\mathbb{K}_m(\operatorname{Proj} X)$ commute with coproducts, and May's axiom (TC3) holds by the previous proposition, so we conclude that Theorem C.1 applies to $\mathbb{K}_m(\operatorname{Proj} X)$. The same is true of $\mathbb{D}(\mathfrak{Qco} X)$ without the noetherian hypothesis.

Bibliography

[AB56] Maurice Auslander and David A. Buchsbaum, *Homological dimension in Noetherian rings*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 36–38.

[AB69] Maurice Auslander and Mark Bridger, *Stable module theory*, Memoirs of the American Mathematical Society, No. 94, American Mathematical Society, Providence, R.I., 1969.

[AB89] Maurice Auslander and Ragnar-Olaf Buchweitz, *The homological theory of maximal Cohen-Macaulay approximations*, Mém. Soc. Math. France (N.S.) (1989), no. 38, 5–37, Colloque en l'honneur de Pierre Samuel (Orsay, 1987).

[AM02] Luchezar L. Avramov and Alex Martsinkovsky, *Absolute, relative, and Tate cohomology of modules of finite Gorenstein dimension*, Proc. London Math. Soc. (3) **85** (2002), no. 2, 393–440.

[AM93] J. Asensio Mayor and J. Martínez Hernández, *On flat and projective envelopes*, J. Algebra **160** (1993), no. 2, 434–440.

[AJL97] Leovigildo Alonso Tarrío, Ana Jeremías López, and Joseph Lipman, *Local homology and cohomology on schemes*, Ann. Sci. École Norm. Sup. (4) **30** (1997), no. 1, 1–39.

[AJPV07] Leovigildo Alonso Tarrío, Ana Jeremías López, Marta Pérez Rodríguez, and María J. Vale Gonsalves, *The derived category of quasi-coherent sheaves and axiomatic stable homotopy*, 2007, arXiv:0706.0493v1.

[AJS00] Leovigildo Alonso Tarrío, Ana Jeremías López, and María José Souto Salorio, *Localization in categories of complexes and unbounded resolutions*, Canad. J. Math. **52** (2000), no. 2, 225–247.

[Aus86] Maurice Auslander, *Rational singularities and almost split sequences*, Trans. Amer. Math. Soc. **293** (1986), no. 2, 511–531.

[BBD82] Alexander Beĭlinson, Joseph Bernstein, and Pierre Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171.

[BCR97] David J. Benson, Jon F. Carlson, and Jeremy Rickard, *Thick subcategories of the stable module category*, Fund. Math. **153** (1997), no. 1, 59–80.

[BD06] Igor Burban and Yuriy Drozd, *Derived categories for nodal rings and projective configurations*, Noncommutative algebra and geometry, Lect. Notes Pure Appl. Math., vol. 243, Chapman & Hall/CRC, Boca Raton, FL, 2006, pp. 23–46.

[BEB01] Ladislav Bican, Robert El Bashir, and Edgar Enochs, *All modules have flat covers*, Bull. London Math. Soc. **33** (2001), no. 4, 385–390.

[BGS87] Ragnar-Olaf Buchweitz, Gert-Martin Greuel, and Frank-Olaf Schreyer, *Cohen-Macaulay modules on hypersurface singularities. II*, Invent. Math. **88** (1987), no. 1, 165–182.

[BK81] Jean-Luc Brylinski and Masaki Kashiwara, *Kazhdan-Lusztig conjecture and holonomic systems*, Invent. Math. **64** (1981), no. 3, 387–410.

[BKR01] Tom Bridgeland, Alastair King, and Miles Reid, *The McKay correspondence as an equivalence of derived categories*, J. Amer. Math. Soc. **14** (2001), no. 3, 535–554.

[BN93] Marcel Bökstedt and Amnon Neeman, *Homotopy limits in triangulated categories*, Compositio Math. **86** (1993), no. 2, 209–234.

[BO02] A. Bondal and D. Orlov, *Derived categories of coherent sheaves*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002) (Beijing), Higher Ed. Press, 2002, pp. 47–56.

[Bou79] A. K. Bousfield, *The localization of spectra with respect to homology*, Topology **18** (1979), no. 4, 257–281.

[Bri06] Tom Bridgeland, *Derived categories of coherent sheaves*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 563–582.

[Bro90] Michel Broué, *Isométries parfaites, types de blocs, catégories dérivées*, Astérisque (1990), no. 181-182, 61–92.

[Buc87] Ragnar-Olaf Buchweitz, *Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings*, Unpublished (1987), 155pp.

[Chr98] J. Daniel Christensen, *Ideals in triangulated categories: phantoms, ghosts and skeleta*, Adv. Math. **136** (1998), no. 2, 284–339.

[Chr00] Lars Winther Christensen, *Gorenstein dimensions*, Lecture Notes in Mathematics, vol. 1747, Springer-Verlag, Berlin, 2000.

[CKN01] J. Daniel Christensen, Bernhard Keller, and Amnon Neeman, *Failure of Brown representability in derived categories*, Topology **40** (2001), no. 6, 1339–1361.

[Con00] Brian Conrad, *Grothendieck duality and base change*, Lecture Notes in Mathematics, vol. 1750, Springer-Verlag, Berlin, 2000.

[CR00] Jon F. Carlson and Raphaël Rouquier, *Self-equivalences of stable module categories*, Math. Z. **233** (2000), no. 1, 165–178.

[Din96] Nanqing Ding, *On envelopes with the unique mapping property*, Comm. Algebra **24** (1996), no. 4, 1459–1470.

[Dro04] Yuriy A. Drozd, *Vector bundles and Cohen-Macaulay modules*, Representations of finite dimensional algebras and related topics in Lie theory and geometry, Fields Inst. Commun., vol. 40, Amer. Math. Soc., Providence, RI, 2004, pp. 189–222.

[EE05a] Edgar Enochs and Sergio Estrada, *Relative homological algebra in the category of quasi-coherent sheaves*, Adv. Math. **194** (2005), no. 2, 284–295.

[EE05b] ———, *Relative homological algebra in the category of quasi-coherent sheaves*, Adv. Math. **194** (2005), no. 2, 284–295.

[EEGO04] Edgar Enochs, S. Estrada, J. R. García-Rozas, and L. Oyonarte, *Flat cotorsion quasi-coherent sheaves. Applications*, Algebr. Represent. Theory **7** (2004), no. 4, 441–456.

[EGR98] Edgar E. Enochs and J. R. García Rozas, *Flat covers of complexes*, J. Algebra **210** (1998), no. 1, 86–102.

[EJ85] Edgar E. Enochs and Overtoun M. G. Jenda, *Balanced functors applied to modules*, J. Algebra **92** (1985), no. 2, 303–310.

[EJ95] ———, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), no. 4, 611–633.

[EJ00] ———, *Relative homological algebra*, de Gruyter Expositions in Mathematics, vol. 30, Walter de Gruyter & Co., Berlin, 2000.

[EJ06] ———, *Gorenstein homological algebra*, Abelian groups, rings, modules, and homological algebra, Lect. Notes Pure Appl. Math., vol. 249, Chapman & Hall/CRC, Boca Raton, FL, 2006, pp. 75–86.

[Ene07] Viviana Ene, *Maximal Cohen-Macaulay modules over hypersurface rings*, An. St. Univ. Ovidius Constanta **15** (2007), no. 1, 75–90.

[Eno81] Edgar E. Enochs, *Injective and flat covers, envelopes and resolvents*, Israel J. Math. **39** (1981), no. 3, 189–209.

[Esn85] Hélène Esnault, *Reflexive modules on quotient surface singularities*, J. Reine Angew. Math. **362** (1985), 63–71.

[Fre07] Edward Frenkel, *Lectures on the Langlands program and conformal field theory*, Frontiers in number theory, physics, and geometry. II, Springer, Berlin, 2007, pp. 387–533.

[SGA6] Alexander Grothendieck, P. Berthelot, and L. Illusie, *Théorie des intersections et théorème de Riemann-Roch*, Lecture Notes in Mathematics, vol. 255, Springer-Verlag, 1971.

[Hap88] Dieter Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, vol. 119, Cambridge University Press, Cambridge, 1988.

[Har66] Robin Hartshorne, *Residues and duality*, Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966.

[Har67] ———, *Local cohomology*, A seminar given by A. Grothendieck, Harvard University, Fall, vol. 1961, Springer-Verlag, Berlin, 1967.

[Har77] ———, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52.

[HdB04] Lutz Hille and Michel Van den Bergh, *Fourier-Mukai transforms*, 2004, arXiv:math.AG/0402043.

[HJ07] Henrik Holm and Peter Jørgensen, *Compactly generated homotopy categories*, Homology, Homotopy Appl. **9** (2007), no. 1, 257–274.

[Hol04] Henrik Holm, *Gorenstein derived functors*, Proc. Amer. Math. Soc. **132** (2004), no. 7, 1913–1923

[Hov01] Mark Hovey, *Model category structures on chain complexes of sheaves*, Trans. Amer. Math. Soc. **353** (2001), no. 6, 2441–2457.

[HPS97] Mark Hovey, John H. Palmieri, and Neil P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. **128** (1997), no. 610, x+114.

[IK06] Srikanth Iyengar and Henning Krause, *Acyclicity versus total acyclicity for complexes over Noetherian rings*, Doc. Math. **11** (2006), 207–240.

[Jør05] Peter Jørgensen, *The homotopy category of complexes of projective modules*, Adv. Math. **193** (2005), no. 1, 223–232.

[Kas84] Masaki Kashiwara, *The Riemann-Hilbert problem for holonomic systems*, Publ. Res. Inst. Math. Sci. **20** (1984), no. 2, 319–365.

[KN02] Bernhard Keller and Amnon Neeman, *The connection between May's axioms for a triangulated tensor product and Happel's description of the derived category of the quiver D_4*, Doc. Math. **7** (2002), 535–560.

[Knö87] Horst Knörrer, *Cohen-Macaulay modules on hypersurface singularities. I*, Invent. Math. **88** (1987), no. 1, 153–164.

[Kon95] Maxim Kontsevich, *Homological algebra of mirror symmetry*, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994) (Basel), Birkhäuser, 1995, pp. 120–139.

[Kra01] Henning Krause, *On Neeman's well generated triangulated categories*, Doc. Math. **6** (2001), 121–126.

[Kra02] ———, *A Brown representability theorem via coherent functors*, Topology **41** (2002), no. 4, 853–861.

[Kra05] ———, *The stable derived category of a Noetherian scheme*, Compos. Math. **141** (2005), no. 5, 1128–1162.

[Laz64] Daniel Lazard, *Sur les modules plats*, C. R. Acad. Sci. Paris **258** (1964), 6313–6316.

[Lip] Joseph Lipman, *Notes on derived categories and derived functors*, Available from: http://www.math.purdue.edu/~lipman/Duality.pdf.

[Mat58] Eben Matlis, *Injective modules over Noetherian rings*, Pacific J. Math. **8** (1958), 511–528.

[May01] J. P. May, *The additivity of traces in triangulated categories*, Adv. Math. **163** (2001), no. 1, 34–73.

[Miy07] Jun-Ichi Miyachi, *Compact objects in stable module categories*, Arch. Math. **89** (2007), no. 1, 47–51.

[MVW06] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, vol. 2, American Mathematical Society, Providence, RI, 2006.

[Nee92] Amnon Neeman, *The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 5, 547–566.

[Nee96] ———, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability*, J. Amer. Math. Soc. **9** (1996), no. 1, 205–236.

[Nee01a] ———, *On the derived category of sheaves on a manifold*, Doc. Math. **6** (2001), 483–488.

[Nee01b] ———, *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001.

[Nee06a] ———, *The homotopy category of flat modules, and Grothendieck duality*, Unpublished (2006).

[Nee06b] ———, *A non-commutative generalisation of Thomason's localisation theorem*, Non-commutative localization in algebra and topology, London Math. Soc. Lecture Note Ser., vol. 330, Cambridge Univ. Press, Cambridge, 2006, pp. 60–80.

[Nee06c] ———, *Some adjoints in homotopy categories*, Unpublished (2006).

[Nee07] ———, *An infinite version of homological mirror symmetry*, Real and complex singularities, proceedings of the Australian-Japanese workshop, World Scientific, 2007.

[Orl04] D. O. Orlov, *Triangulated categories of singularities and D-branes in Landau-Ginzburg models*, Tr. Mat. Inst. Steklova **246** (2004), no. Algebr. Geom. Metody, Svyazi i Prilozh., 240–262.

[Ric89] Jeremy Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra **61** (1989), no. 3, 303–317.

[Ric97] ———, *Idempotent modules in the stable category*, J. London Math. Soc. (2) **56** (1997), no. 1, 149–170.

[Rou03] Raphael Rouquier, *Dimensions of triangulated categories*, arXiv:math.CT/0310134.

[Ser56] Jean-Pierre Serre, *Sur la dimension homologique des anneaux et des modules noethériens*, Proceedings of the international symposium on algebraic number theory, Tokyo & Nikko, 1955 (Tokyo), Science Council of Japan, 1956, pp. 175–189.

[Spa88] N. Spaltenstein, *Resolutions of unbounded complexes*, Compositio Math. **65** (1988), no. 2, 121–154.

[Ste75] Bo Stenström, *Rings of quotients*, Springer-Verlag, New York, 1975, Die Grundlehren der Mathematischen Wissenschaften, Band 217, An introduction to methods of ring theory.

[TT90] R. W. Thomason and Thomas Trobaugh, *Higher algebraic K-theory of schemes and of derived categories*, The Grothendieck Festschrift, Vol. III, Progr. Math., vol. 88, Birkhäuser Boston, Boston, MA, 1990, pp. 247–435.

[Ver77] Jean-Louis Verdier, *Catégories dérivées. quelques résultats (etat 0)*, Sémin. Géom. algébr. Bois-Marie, SGA4.5, Lecture Notes in Math. **569** (1977), 262–311.

[Ver96] ———, *Des catégories dérivées des catégories abéliennes*, Astérisque (1996), no. 239, xii+253 pp. (1997), With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis.

[Wei94] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.

[Xu96] Jinzhong Xu, *Flat covers of modules*, Lecture Notes in Mathematics, vol. 1634, Springer-Verlag, Berlin, 1996.

[Yos90] Yuji Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Mathematical Society Lecture Note Series, vol. 146, Cambridge University Press, Cambridge, 1990.